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3. Some classical examples

The most important classes of geometries, historically speaking, are the following: chain groups, function spaces, algebraic extensions of fields, coverings, Wille incidence geometries, graphs and simplicial geometries. As a preliminary to the discussion of these examples, let us look at a few small but otherwise perfectly general geometries. In the next Section, we shall discuss some examples arising in applied mathematics.

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3.1. Small geometries

Given a five-element set, how many different geometries may be constructed upon it? It will turn out that all such geometries can be represented by sets of points in an affine space of dimension no greater than 4 when the closure of a set is the ordinary linear span, so we shall draw them as such. First, there is the unique geometry of rank 2: five points on a line.

FIGURE 3.1

A rank 3 geometry on five points may have at most one line of four points, and at most two lines of three points. There are four possibilities:

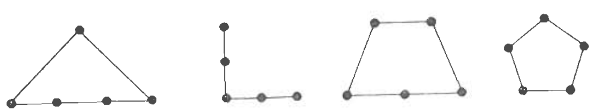

FIGURE 3.2

A rank 4 geometry on five points has at most one nontrivial flat, either a three-point line, a four-point plane, or five points in general position on a flat of rank 4.

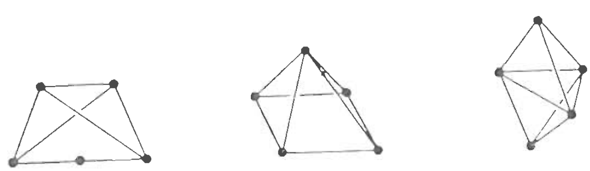

FIGURE 3.3

If the geometry on five points has rank 5, the points must be in general position in a space of rank 5.

A similar count of the number $g_{n,k}$ of essentially different geometries of rank k on an n-element set, $n = 1, \cdots, 8$, yields the following tabulation. Let g_n be the total $g_{n,1} + \cdots + g_{n,n}$. Then the recursion

$$g_{n+1} = (g_n)^{3/2}$$

seems approximately correct, on the basis of this data alone. This would suggest that there are some thirty thousand essentially different geometries on a nine-element set.

TABLE 1. TABULATION OF $g_{n,k}$

	8 pts.	7 pts.	6 pts.	5 pts.	4 pts.	3 pts.	2 pts.	1 pt.
rank 1	.	.	.	.	.	.	.	1
2	1	1	1	1	1	1	1	
3	68	23	9	4	2	1		
4	617	49	11	3	1			
5	217	22	4	1				
6	40	5	1					
7	6	1						
8	1							
total	950	101	26	9	4	2	1	1

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3.2. CHAIN GROUPS

Let M be a module over a commutative integral domain R. The set, upon which we shall define a closure relation, will be an arbitrary finite subset S of the module M.

For any subset $A \subseteq S$, let $\bar{A}$ be the set

$$\bar{A} = \{x \in S;\ \text{some nonzero } R\text{-multiple of } x \text{ is expressible as a finite linear combination of elements of } A, \text{ with coefficients in } R\}.$$

PROPOSITION 3.1. The subset S, furnished with the relation $A \to \bar{A}$, is a pregeometry.

Proof: For any subset $A \subseteq S$, $A \subseteq \bar{A}$. Assume $A \subseteq \bar{B}$ and $x \in \bar{A}$. Then for some nonzero $r \in R$, and coefficients $r_i \in R$, $rx = r_1 m_1 + \cdots + r_k m_k$, with $m_i \in A$. Since the elements m_i are in $\bar{B}$, there exist nonzero elements $s_1, \cdots, s_k$ in R such that $s_i m_i$ is a finite linear combination of elements of B. Since R is an integral domain, the product $s_1 \cdots s_n r$ is nonzero. Thus $s_1 \cdots s_n r x$ is a nonzero multiple of x, which is expressible as a finite linear combination of elements of B, because R is commutative. So $x \in \bar{B}$, $\bar{A} \subseteq \bar{B}$, and the function $A \to \bar{A}$ is a closure.

Say that $y \in \overline{A \cup x}$, $y \notin \bar{A}$. Then for some $r \neq 0$,

$$ry = sx + \sum_{i=1}^{n} r_i z_i, \qquad z_i \in A.$$

But $y \notin \bar{A}$ implies $s \neq 0$, so

$$sx = ry + \sum_{i=1}^{n} (-r_i) z_i, \qquad z_i \in A$$

and $x \in \overline{A \cup y}$. ∎

This pregeometry is called the *chain- group* pregeometry $C(S)$. It is by far the most important example of a geometry. The reader is urged to peruse the connection between the exchange property and the "elimination" of a variable in linear algebra that is displayed by the preceding example. This connection embodies the "yoga" of the theory of combinatorial geometry.

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44. But this expresses the fact that, if we take the quaternary

$$Pdx + Qdy + Rdz + Tdt,$$

and express the four discriminoids formed by supposing x, y, z, and t successively to be constant [viz., $\square_2(Q, R, T)$, $\square_2(R, T, P)$, $\square_2(T, P, Q)$, and $\square_2(P, Q, R)$], then those four discriminoids are connected by a linear relation, so that, if three vanish identically, so will the fourth. Now, in order to apply the contents of this paper to the ascertaining whether a given quaternary admits of a single solution containing all the variables, we proceed thus: form $\square_2(P, Q, R)$, and if it vanishes identically form $\square_2(Q, R, T)$, and if that again is null form $\square_2(R, T, P)$. If that too vanishes identically, then we may be sure that $\square_2(T, P, Q)$ also vanishes identically, and that the quaternary is completely integrable. If, however, one of the discriminoids, say $\square_2(P, Q, R)$, does not vanish identically, then seek all the exceptional solutions, discriminoidal or other, of $Pdx + Qdy + Rdz$, which contain x, y, z, and t (*i.e.*, all the four variables). If no such solution can be found, the quaternary has no single solution in all the variables. If such a solution can be found, and reduces to zero the quaternary as well as the ternary, it is the required single solution of the latter. If it does not reduce the quaternary to zero, then the quaternary has no single solution whatever. It should be remarked that any one of the contained ternaries may possibly have more suitable solutions than one, and each one of the solutions of the ternary first tried should be tested by the quaternary. The process is much the same for quinary and higher forms.

Discussion of Two Double Series arising from the Number of Terms in Determinants of Certain Forms. By J. D. H. Dickson, M.A.

[*Read March 13th*, 1879.]

The first double series arises from the number of non-vanishing terms of a determinant of n^2 elements, with one diagonal of r zero-elements.

If $u_{n,r}$ be the number of such terms in a determinant as above described, it is found, by summation, in two different ways, that

$$u_{n,r} = (n-r)\, u_{n-1,r-1} + (r-1)\, u_{n-1,r-2} \quad \text{............ (1)},$$

and

$$u_{n,r} = u_{n,r-1} + u_{n-1,r} \quad \text{.......................... (2)}.$$

From (2), if E, F be two operators operating only on u, and such that E refers to n alone, and F to r alone, then

$$E = EF + 1 \quad \text{.......................... (3)};$$

whence, in combination with (1), various formulæ may be obtained. For example, a formula connecting three determinants of the same order, is

$$(n-r+1)\,u_{n,r} = (n-2r+1)\,u_{n,r-1} + (r-1)\,u_{n,r-2} \;\ldots\ldots\; (4);$$

and one connecting three consecutive numbers on the same diagonal of Table No. 1, is

$$u_{n,r} = (n-1)\,u_{n-1,r-1} + (r-1)\,u_{n-2,r-2} \;\ldots\ldots\ldots\ldots\; (5).$$

From (5), by putting $r=n$, we have

$$u_{n,n} = (n-1)\,(u_{n-1,n-1} + u_{n-2,n-2}) \;\ldots\ldots\ldots\ldots\; (6);$$

a first integral of which is

$$u_{n,n} = n\,u_{n-1,n-1} + (-1)^n \;\ldots\ldots\ldots\ldots\ldots\ldots\; (7).$$

The generating function of (7) is easily found to be

$$u = \frac{e^{-x}}{1-x} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (8).$$

Table No 1 is a table of values of $u_{n,r}$, calculated by means of (2); from which, in the case of $n=7$, $r=3$, we have, as examples of (1), (2), (4), and (7),

$$3216 = 4\,.\,504 + 2\,.\,600,$$
$$3216 = 2790 + 426,$$
$$5\,.\,3216 = 2\,.\,3720 + 2\,.\,4320,$$
$$1854 = 7\,.\,265 - 1.$$

The second double series arises from the number of non-vanishing terms of a determinant of n^2 elements, with two adjacent diagonals of r and $r-1$ zero-elements.

If $v_{n,r}$ be the number of such terms in a determinant as above described, it is found that

$$v_{n,r} = v_{n,r+1} + 2v_{n-1,r} + v_{n-2,r-1} \;\ldots\ldots\ldots\ldots\ldots\; (9).$$

From (9), using the same operators as in the previous case, we have

$$E^2F = E^2F^2 + 2EF + 1 \;\text{ or }\; = (EF+1)^2 \;\ldots\ldots\ldots\; (10).$$

By a process of some length, which does not appear to admit of much simplification, it is also found that

$$v_{n,n} = (n-1)(v_{n-1,n-1} + v_{n-2,n-2}) + v_{n-3,n-3} \;\ldots\ldots\ldots\; (11),$$

a first integral of which is

$$(n-1)\,v_{n,n} = (n^2-n-1)\,v_{n-1,n-1} + n\,v_{n-2,n-2} - (-1)^n\,2 \;\ldots\; (12).$$

Table No. 2 is a table of values of $v_{n,r}$, calculated by means of (9); from which, in the case of $n=8$, $r=5$, we have, as examples of (9), (11), and (12),

$$11274 = 8756 + 2\,.\,1168 + 182,$$
$$5413 = 7\,(675 + 96) + 16,$$
$$7\,.\,5413 = 55\,.\,675 + 8\,.\,96 - 2.$$

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No of terms in certain determinants

TABLE No. 1. VALUES OF $u_{n,r}$, AS FAR AS $n=10$, $r=10$.

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n \ r =	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	0									
2	2	1	1								
3	6	4	3	2							
4	24	18	14	11	9						
5	120	96	78	64	53	44					
6	720	600	504	426	362	309	265				
7	5040	4320	3720	3216	2790	2428	2119	1854			
8	40320	35280	30960	27240	24024	21234	18806	16687	14833		
9	362880	322560	287280	256320	229080	205056	183822	165016	148329	133496	
10	3628800	3265920	2943360	2656080	2399760	2170680	1965624	1781802	1616786	1463457	1334961

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TABLE No. 2. VALUES OF $v_{n,r}$, AS FAR AS $n=10$, $r=10$.

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Σ = 47922 ditto

$\boxed{1}$ $\boxed{2}$ $\boxed{1}$ add these to get

n \ r =	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	0									
2	2	1	0								
3	6	4	1	1							
4	24	18	8	5	3						
5	120	96	54	34	23	16					
6	720	600	384	258	182	131	96				
7	5040	4320	3000	2136	1566	1168	883	675			
8	40320	35280	25920	19320	14664	11274	8756	6859	5413		
9	362880	322560	246960	190800	149160	117696	93582	74902	60301	48800	
10	3628800	3265920	2580480	2051280	1643760	1326120	1076064	877626	719066	591605	488592

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$n^2 n!$

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