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MATHEMATICS

SELF-COMPLEMENTARY AND SELF-CONVERSE ORIENTED GRAPHS

BY

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Abstract

DE BRUIJN [1] applied his generalization of Pólya's fundamental theorem to provide an outline of a general method for enumerating self-complementary structures. This was used by READ [8] to carry out in detail the enumeration of self-complementary graphs and digraphs. Suitable modifications of the same scheme gave HARARY and PALMER [4], [5] the basic clue for their enumeration of self-converse digraphs. In this paper we extend these results to obtain the formula for the number of self-complementary oriented graphs on n points and the generating function for self-converse oriented graphs in terms of the number of lines.

1. SELF-COMPLEMENTARY ORIENTED GRAPHS

The generating functions for self-complementary graphs and digraphs are respectively Z_{G_n} (0, 2, 0, 2, ...) and Z_{D_n} (0, 2, 0, 2, ...) where G_n is the configuration group $S_n^{(2)}$ for graphs and D_n is the configuration group $S_n^{[2]}$ for digraphs. When we write analogously Z_{Q_n} (0, 2, 0, 2, ...) for the generating function for self-complementary oriented graphs, Q_n being the configuration group as defined in [3], we come across a difficulty which is deeper than a mere notational one. For example in [3] Harary observes in one place (p. 221) that Q_n is a permutation group of degree $p(p-1)/2$ whereas in computing the contribution to oriented graphs certain cycles from some permutations are deleted and the resulting formula $Z(Q_n)$ is not a proper cycle index in the usually accepted sense of the term. Since we see no obvious way of overcoming this difficulty, instead of deriving the generating function for self-complementary oriented graphs by applying de Bruijn's theorem [1] with Q_n acting on $X^{(2)}$ and S_2 on $\{0, 1\}$, we start with Read's result for self-complementary digraphs and pick out the oriented graphs from these by eliminating all those permutations of D_n which give rise to non-oriented graphs.

The cycle index of D_n was first obtained by Pólya and was described by HARARY [2] in his famous expository article dealing with graphical enumeration and is given as

$$Z_{D_n}(f_1, f_2, \ldots f_n, \ldots) = \frac{1}{n!} \sum_{\alpha \in S_n} \prod_{\nu=0}^{[\frac{1}{2}(n-1)]} f_{2\nu+1}^{\nu j_{2\nu+1} + (\nu+\frac{1}{2}) j_{2\nu+1}(j_{2\nu+1}-1)}$$

$$\times \prod_{\nu=1}^{[\frac{1}{2}n]} \left(f_r f_{2\nu}^{\nu-1}\right)^{j_{2\nu}} f_{2\nu}^{\nu j_{2\nu}(j_{2\nu}-1)} \prod_{1 \leqslant q < r \leqslant n} f_{<q,r>}^{j_q j_r (q,r)}$$

where (q, r) is the g.c.d and $\langle q, r\rangle$ is the l.c.m. of q and r and α has cycle structure $(1^{j_1}\, 2^{j_2}\, 3^{j_3} \ldots)$. The self-complementary digraphs on n points are obtained from this by putting $f_\nu = 0$ or 2 according as ν is odd or even. READ [8] has shown that this leads to the conclusion that only permutations of S_n with cycle structure $(1^{j_1}\, 2^{j_2}\, 4^{j_4}\, 6^{j_6}\, 8^{j_8}\, 10^{j_{10}} \ldots)$ with $j_1 = 0$ or 1 give non-zero contributions to the expression Z_{D_n} $(0, 2, 0, 2, \ldots)$.

We now make the modifications in this result so that we get self-complementary oriented graphs. We observe that if $\alpha \in S_n$ has a cycle whose length is a multiple of four, the resulting self-complementary digraph is *not* oriented. For example, if (1234) is a cycle of α, the corresponding cycles of the permutation g_α of D_n induced by α are

$$((1,2)(2,3)(3,4)(4,1)),\ ((1,3)(2,4)(3,1)(4,2)),$$
$$((2,1)(3,2)(4,3)(1,4))$$

Since a self-complementary digraph should satisfy the condition $f(d) = \beta f(g_\alpha d)$ where $d \in X^{[2]}$ and β is the cycle of length 2 on the symbols 0, 1, the presence of the middle cycle requires that either $f(1,3) = f(3,1) = 1$ or $f(2,4) = f(4,2) = 1$. In either case we get a non-oriented digraph. Thus, only permutations of S_n with cycle structure of the form

$$(1^{j_1}\, 2^{j_2}\, 6^{j_6}\, 10^{j_{10}}\, 14^{j_{14}} \ldots)$$

give nonzero contribution to the generating function.

The main observation in computing the contribution from a typical permutation of the above form is that the contribution gets halved from the corresponding contribution for self-complementary digraphs. The detailed contributions are given below with illustrations:

(i) A cycle of length $\nu = 4a + 2$ in $\alpha \in S_n$ gives rise to $\nu - 1$ cycles of length ν in g_α but the contribution from such a cycle to the expression, in which the substitution $f_\nu = 0$ or 2 according as ν is odd or even has to be made, comes out to be only $\dfrac{\nu - 2}{2} + 1$ cycles of length ν.

Example. $\alpha = (123456)$ gives rise to five cycles of g_α.

$$((1,2)(2,3)(3,4)(4,5)(5,6)(6,1))\quad ((2,1)(3,2)(4,3)(5,4)(6,5)(1,6))$$
$$((1,3)(2,4)(3,5)(4,6)(5,1)(6,2))\quad ((3,1)(4,2)(5,3)(6,4)(1,5)(2,6))$$
$$((1,4)(2,5)(3,6)(4,1)(5,2)(6,3))$$

The pairs of cycles in the first two rows may be called converse pairs. If digraphs were permitted, each such pair will correspond to 4 possible combinations obtained from

$$f(1,2) = \begin{cases}0\\1\end{cases} \text{ and } f(2,1) = \begin{cases}0\\1\end{cases}$$

But to obtain an oriented graph the contributions $f(1,2) = f(2,1) = 1$ and $f(1,2) = f(2,1) = 0$ have to be disallowed. The last cycle does not give rise to any such difficulty. Thus the contribution from this cycle to the cycle index for self-complementary oriented graphs is $f_6^{j_6}$ where $j_6 = \frac{6-2}{2} + 1$. The total contribution from individual cycles of length ν will therefore be

$$f_\nu^{((\nu-2)/2+1)\, j_\nu}$$

(ii) A pair of distinct cycles of length ν gives rise to $2\nu^2$ point pairs (i, j) which arrange themselves into 2ν cycles of length ν each. These being pairs of converse cycles, the contribution is f_ν^ν. The total contribution from such pairs is therefore $f_\nu^{j_\nu(j_\nu-1)\,\nu/2}$.

Example. (12) (34) gives rise to the following converse pairs.

((1,3)(2,4)) ((3,1)(4,2)); ((1,4)(2,3)) ((4,1)(3,2))

(iii) Any two cycles of unequal lengths $q = 4a+2$ and $r = 4b+2$ give rise to (q, r) pairs of converse cycles of length $\langle q, r \rangle$ each and the total contribution from such cycle pairs is $f_{\langle q,r \rangle}^{j_q j_r (q,r)}$.

(iv) The j_1 $(=0$ or $1)$ trivial cycles do not give any individual contribution and their contribution in pairs with other cycles can be obtained by the formula in (iii).

Thus we have the following lemma:

Lemma 1. The generating function for self-complementary oriented graphs is

$$\bar{O}_n = \frac{1}{n!} \sum_{\alpha \in S_n} \prod_{\nu \in N} f_\nu^{((\nu-2)/2+1)\, j_\nu + \nu/2\, (j_\nu - 1)\, j_\nu} \prod_{\substack{q<r \\ \in N'}} \prod f_{\langle q,r \rangle}^{(q,r)\, j_q\, j_r} \tag{1}$$

where $N = \{2, 6, 10, 14, \ldots\}$ and $N' = \{1, 2, 6, 10, 14, \ldots\}$.

Since every cycle in the formula of lemma 1 is of even length the result of putting $f_\nu = 0$ or 2 according as ν is odd or even leads to the following theorem:

Theorem 1. The number of self-complementary oriented graphs is

$$Y_n = \frac{1}{n!} \sum_{\alpha \in S_n} 2^{\bar{O}_n(\alpha)}$$

where

$$\bar{O}_n(\alpha) = \sum_{\nu \in N} \frac{\nu}{2} j_\nu^2 + \sum_{\substack{q<r \\ \in N'}} \sum (q, r)\, j_q j_r. \tag{2}$$

The numbers of self-complementary oriented graphs with up to 10 points are given in the following table:

n	3	4	5	6	7	8	9	10
Y_n	2	2	8	12	88	176	2752	8784

Please enter 1 ✓

2785

Since self-complementary oriented graphs must necessarily have $n(n-1)/2$ lines and these must be tournaments, that is, complete oriented graphs, it is not difficult to pick them out from digraphs. For example, though there are 1670 digraphs on 5 points with 10 lines [2, p. 453] only 12 of them are oriented graphs [3, p. 224]. These can be easily drawn systematically from the complete oriented graphs on 4 points with 6 lines (Graphs 45–48 in HARARY et al. [6]) by adding an additional point and 4 oriented lines. The eight self-complementary oriented graphs with 5 points and 10 lines are given below.

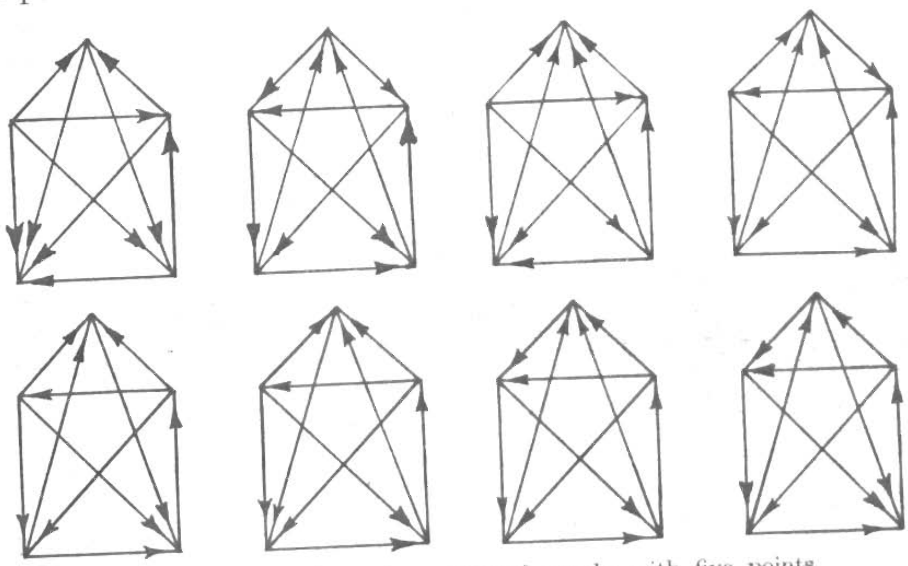

Fig. 1. Self-complementary oriented graphs with five points.

Since every self-complementary oriented graph is a complete oriented graph and the converse and complement of a complete oriented graph are the same, the self-dual oriented graphs are the same as the self-complementary oriented graphs. The enumeration obtained above therefore partially settles problem 6 in Harary's latest list of unsolved problems [7, p. 30].

2. SELF-CONVERSE ORIENTED GRAPHS

The generating function for self-converse digraphs has been obtained in [5] as

$$F(S_n{}^{*}s_2, 1+x, 1+x^2, \ldots) = \frac{1}{n!} \sum_{\alpha \in S_n} I(\alpha, 1+x, 1+x^2, \ldots)$$

where

$$I(\alpha) = \prod_{k=1}^{n} a_{\langle 2,k\rangle}^{(2,k)\, k\binom{j_k}{2}} \prod_{1 \leqslant r < s \leqslant n} a_{\langle 2,\langle r,s\rangle\rangle}^{(2,\langle r,s\rangle)\,(r,s)\, j_r j_s}$$

$$\times \prod_{k \text{ odd}} a_{2k}^{j_k((k-1)/2)} \prod_{k \text{ even}} a_k^{(k-2)\, j_k}\, a_{k/2}^{\eta(k)\, 2j_k}\, a_k^{(1-\eta(k))\, j_k}$$

where $\eta(k) = 1$ if $k/2$ is odd and 0 otherwise.

To select the oriented graphs from these, one has to delete those cycles of $g_\alpha \in S_n{}^{*S_2}$ in which both the pairs of points (i, j) and (j, i) occur. This is so because in enumerating self-converse digraphs the restriction imposed on f is $f(d) = f(g_\alpha d)$ so that (i, j) and (j, i) will take the same weight 1 if they appear in the same cycle. Observing further that the action of $g_\alpha \in S_n{}^{*S_2}$ on $X^{[2]}$ induced by $\alpha \in S_n$ is given by $g_\alpha\ (i, j) = (\alpha j, \alpha i)$ we arrive at the following conclusions.

(i) Odd cycles of α give rise to inadmissible cycles of g_α.

Example. (12345) induces the following cycles in g_α.

$$((1,2)(3,2)(3,4)(5,4)(5,1)(2,1)(2,3)(4,3)(4,5)(1,5))$$
$$((1,3)(4,2)(3,5)(1,4)(5,2)(3,1)(2,4)(5,3)(4,1)(2,5)).$$

These cycles are inadmissible because they contain the pairs (i, j) and (j, i) in the same cycle.

(ii) Any two odd cycles of α give rise to inadmissible cycles of g_α.

Example. (1) (234) gives rise to the cycle

$$((1,2)(3,1)(1,4)(2,1)(1,3)(4,1)).$$

In general (i, j) and (j, i) occur with a separation of $\langle p, q\rangle$ elements, where p and q are the lengths of the two odd cycles. It can be verified that this holds even when $p = q$.

(iii) An even cycle of α, whose length is a multiple of four, gives rise to one inadmissible cycle in g_α while the other cycles form converse pairs.

Example. (1234) → ((1,2)(3,2)(3,4)(1,4)) ((2,1)(2,3)(4,3)(4,1))
((1,3)(4,2)(3,1)(2,4)).

(iv) Even cycles of α whose lengths are not multiples of 4 give rise to admissible cycles of g_α which pair off into converse cycles.

Example. (123456) → ((1,2)(3,2)(3,4)(5,4)(5,6)(1,6))
((2,1)(2,3)(4,3)(4,5)(6,5)(6,1))
((1,3)(4,2)(3,5)(6,4)(5,1)(2,6))
((3,1)(2,4)(5,3)(4,6)(1,5)(6,2))
((1,4)(5,2)(3,6)) ((4,1)(2,5)(6,3)).

(v) Cycle pairs of α whose lengths are unequal and are both even or one even and one odd give rise to admissible cycles of g_α which pair off into converse cycles.

If $I'(\alpha)$ denotes the expression corresponding to $I(\alpha)$ for self-converse oriented graphs, the above considerations lead to the following calculations.

(1) The contribution to $I'(\alpha)$ from individual even cycles is

$$\prod_{k \in M} a_k^{(k-2)/2\, j_k} \prod_{k \in N} a_k^{(k-2)/2\, j_k} \; a_{k/2}^{j_k}$$

where

$$M = \{4, 8, 12, \ldots\}$$

and

$$N = \{2, 6, 10, \ldots\}$$

(2) The contribution to $I'(\alpha)$ from pairs of even cycles of same length is

$$\prod_{k \in M \cup N} a_k^{k\, j_k (j_k - 1)/2}.$$

Combining (1) and (2) we have the following lemma:

Lemma 2. The contribution to $I'(\alpha)$ from all even cycles is

$$\prod_{k \text{ even}} a_k^{j_k (k\, j_k - 2)/2} \; a_{k/2}^{j_k\, \eta(k)}$$

where

$$\eta(k) = \begin{cases} 1 & \text{if } k \in N \\ 0 & \text{if } k \in M \end{cases}$$

(3) Pairs of cycles of α of unequal lengths give the contribution

$$\prod_{\substack{1 \leqslant r < s \leqslant n \\ r \text{ and } s \text{ not} \\ \text{both odd.}}} a_{\langle r, s \rangle}^{(r, s)\, j_r\, j_s}$$

Hence we have

Lemma 3.

(3) $$I'(\alpha) = \prod_{k \text{ even}} a_k^{j_k (k\, j_k - 2)/2} \; a_{k/2}^{j_k\, \eta(k)} \prod_{\substack{1 \leqslant r < s \leqslant n \\ r \text{ and } s \text{ not} \\ \text{both odd.}}} a_{\langle r, s \rangle}^{(r, s)\, j_r\, s}$$

Finally we have the following theorem:

Theorem 2. The generating function for self-converse oriented graphs is

(4) $$0'_n(x) = \frac{1}{n!} \sum_{\alpha \in S_n} I'(\alpha, 1 + 2x, 1 + 2x^2, 1 + 2x^3, \ldots)$$

The generating functions for these graphs with upto 6 points have been computed and are given below:

n	$0'_n(x)$
2	$1 + x$
3	$1 + x + x^2 + 2x^3$
4	$1 + x + 2x^2 + 4x^3 + 4x^4 + 4x^5 + 2x^6$
5	$1 + x + 2x^2 + 5x^3 + 9x^4 + 14x^5 + 17x^6 + 18x^7 + 19x^8 + 8x^9 + 8x^{10}$
6	$1 + x + 2x^2 + 6x^3 + 13x^4 + 27x^5 + 45x^6 + 72x^7 + 104x^8 + 123x^9$ $+ 136x^{10} + 112x^{11} + 104x^{12} + 58x^{13} + 32x^{14} + 12x^{15}.$

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