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SEDLÁČEK'S CONJECTURE ON DISJOINT SOLUTIONS OF $X+Y = Z$

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SEDLÁČEK'S CONJECTURE ON DISJOINT SOLUTIONS OF $X+Y = Z$

Richard K. Guy[1]

In memory of Leo Moser, friend, inspiration, and reason for coming to Alberta.

We wish to partition the numbers from 1 to n into as many disjoint triples as possible, each triple to satisfy the equation $x+y=z$. It is clear that we cannot achieve more than $[n/3]$ triples, where brackets denote "greatest integer not greater than". Suppose that $n = 3k$ and that k triples can be found. Adding the k equations, we see that the total on either side is $\frac{3}{4}k(3k+1)$, i.e. one half of the sum of the first $3k$ numbers. This is not an integer when $n \equiv 6$ or 9, mod 12. Sedláček conjectured [2] that in all other cases $[n/3]$ triples could be found and verified it for $n \leq 24$. We give a proof for all appropriate values of n.

The problem bears a number of similarities to the combinatorial part of the Ringel-Youngs proof [1,3,4] of the Heawood conjecture. This is not surprising since Youngs developed Gustin's method of using a trivalent "current graph" to the edges of which were attached distinct integers which satisfied Kirchhoff's first law, which is the equation of the title. Some of the similarities are

(a) that the solution falls naturally into 12 cases, the residue classes mod 12 to which n may belong; these tend to subdivide because of further parity considerations,

(b) that although solutions exist which follow an infinite pattern within the residue class, there are sometimes irregularities for small

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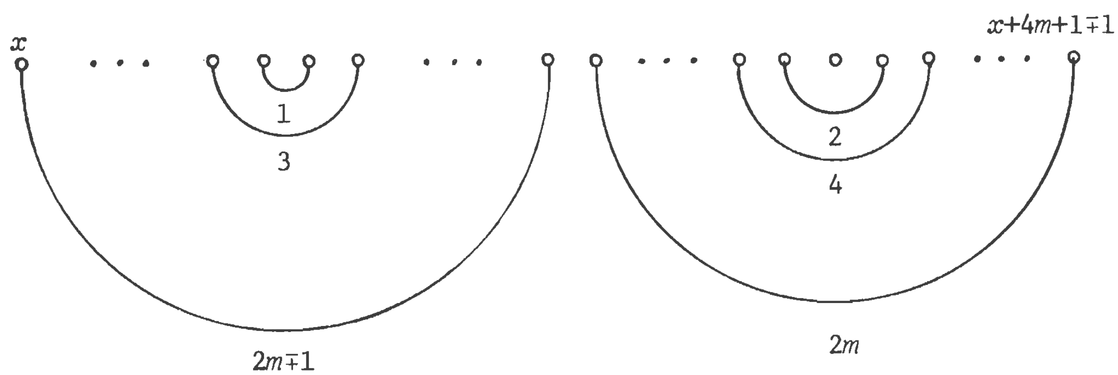
values of n.

(c) that the number of solutions increases exponentially with n, and it is an open question and probably a difficult one: how many solutions are there? (Compare the problem of finding the number of non-isomorphic embeddings of the complete graph on n vertices in a surface of appropriate genus, $\{(n-3)(n-4)/12\}$, where braces denote the post-office function, "least integer not less than".)

(d) that we may use "coil diagrams" of the kind used by Youngs; specifically, if $n = 3k+1$,



the row of nodes represents the consecutive integers from $x = 2m+1$ to $6m+1$, or from $x = 2m+2$ to $6m+4$ according as the upper or lower signs are taken. The integer $5m+1$ (resp. $5m+4$) does not feature in a triple. Each of the $k = 2m$ (resp. $2m+1$) triples is given by a pair of nodes which are connected by a semicircle, and its diameter.

If $n = 3k+2$ we can omit the number $3k+2$ and use the construction just given. The remaining cases, $n = 3k$, where $k \equiv 0$ or 1, mod 4, are dealt with by means of explicit constructions, though we have only been able to do this by subdivision into eight cases for n, mod 48.

$\underline{n = 12r}$. The $4r$ values of z are $[10r+1, 12r] \cup [6r+1, 8r+1] \setminus \{7r+1\}$, i.e., the integers from $10r+1$ through $12r$, and from $6r+1$ through $8r+1$, omitting $7r+1$.

Form the $r-1$ triples $(2i, 6r-i, 6r+i)$, $1 \leq i \leq r-1$,

the $2r-1$ triples $(2i-1, 10r-i+1, 10r+i)$, $1 \leq i \leq 2r-1$

and the two triples $(5r-1, 7r+1, 12r)$ and $(2r, 6r, 8r)$.

There remain r triples to be formed with $z \in \{7r\} \cup [7r+2, 8r-1] \cup \{8r+1\}$ and $x, y \in \{2r+2, 2r+4, 2r+6, \ldots, 4r-2\} \cup [4r-1, 5r-2] \cup \{5r\}$.

$\underline{r = 2s,\ n = 24s}$. Form the s triples $(4s+4i+2, 10s-2i-1, 14s+2i+1)$, $1 \leq i \leq s$, and for

$\underline{s = 2t},\ \underline{n = 48t}$, the triples $(8t+2, 20t, 28t+2)$,

$(8t+8i, 20t-4i+2, 28t+4i+2)$, $1 \leq i \leq t-1$ and

$(8t+8i-4, 20t-4i, 28t+4i-4)$, $1 \leq i \leq t$, or, for

$\underline{s = 2t+1},\ \underline{n = 48t+24}$, the triples $(8t+6, 20t+8, 28t+14)$,

$(8t+8i, 20t-4i+14, 28t+4i+14)$, $1 \leq i \leq t$ and

$(8t+8i+4, 20t-4i+8, 28t+4i+12)$, $1 \leq i \leq t$.

$\underline{r = 2s+1,\ n = 24s+12}$. Form the $s-1$ triples $(4s+4i+10, 10s-2i+1, 14s+2i+11)$ $1 \leq i \leq s-1$, together with the three $(4s+4, 10s+3, 14s+7)$, $(4s+8, 10s+1, 14s+9)$ and $(4s+6, 10s+5, 14s+11)$ or the three $(4s+6, 10s+1, 14s+7)$, $(4s+4, 10s+5, 14s+9)$ and $(4s+8, 10s+3, 14s+11)$, and for

$\underline{s = 2t,\ n = 48t+12}$, the triples $(8t+10, 20t, 28t+10)$,

$(8t+8i+4, 20t-4i+6, 28t+4i+10)$, $1 \leq i \leq t-1$ and

$(8t+8i+8, 20t-4i, 28t+4i+8)$, $1 \leq i \leq t-1$, or, for

$\underline{s = 2t+1,\ n = 48t+36}$, the triples $(8t+14, 20t+12, 28t+26)$ if $t > 0$,

$(8t+8i+12, 20t-4i+14, 28t+4i+26)$, $1 \leq i \leq t-1$ and

$(8t+8i+8, 20t-4i+12, 28t+4i+20)$, $1 \leq i \leq t$.

$\underline{n = 12r+3}$. The $4r+1$ values of z are $[10r+4, 12r+3] \cup [6r+3, 8r+4] \setminus \{7r+4\}$. Form the r triples $(2i, 6r-i+2, 6r+i+2)$, $1 \leq i \leq r$,

the $2r-1$ triples $(2i-1, 10r-i+4, 10r+i+3)$, $1 \leq i \leq 2r-1$

and the two triples $(5r-1, 7r+4, 12r+3)$ and $(2r+2, 6r+2, 8r+4)$.

There remain r triples to be formed with $z \in [7r+3, 8r+3] \setminus \{7r+4\}$ and $x, y \in \{2r+4, 2r+6, 2r+8, \ldots, 4r-2\} \cup [4r-1, 5r+1] \setminus \{5r-1\}$.

$\underline{r = 2s,\ n = 24s+3}$. Form the $s-1$ triples $(4s+4i+8, 10s-2i-3, 14s+2i+5)$, $1 \leq i \leq s-1$, together with the two triples $(4s+6, 10s-3, 14s+3)$ and $(4s+4, 10s+1, 14s+5)$, and for

$\underline{s = 2t,\ n = 48t+3}$, the triples $(8t+8, 20t-2, 28t+6)$,

$(8t+8i+2, 20t-4i+4, 28t+4i+6)$, $1 \leq i \leq t-1$ and

$(8t+8i+6, 20t-4i-2, 28t+4i+4)$, $1 \leq i \leq t-1$, or, for

$\underline{s = 2t+1,\ n = 48t+27}$, the triples $(8t+12, 20t+10, 28t+22)$ if $t > 0$,

$(8t+8i+10, 20t-4i+12, 28t+4i+22)$, $1 \leq i \leq t-1$ and

$(8t+8i+6, 20t-4i+10, 28t+4i+16)$, $1 \leq i \leq t$.

$\underline{r = 2s+1,\ n = 24s+15}$. Form the $s-1$ triples $(4s+4i+4, 10s-2i+7, 14s+2i+11)$, $1 \leq i \leq s-1$, and, for s even, the two triples $(8s+3, 8s+7, 16s+10)$ and $(8s+5, 8s+6, 16s+11)$, or, for s odd, the two triples $(8s+3, 8s+5, 16s+8)$ and $(8s+4, 8s+7, 16s+11)$ and, for

$\underline{s = 2t,\ n = 48t+15}$, $(8t+8i+2, 20t-4i+4, 28t+4i+6)$, $1 \leq i \leq t$ and

$(8t+8i-2, 20t-4i+10, 28t+4i+8)$, $1 \leq i \leq t$,

or, for

$\underline{s = 2t+1,\ n = 48t+39}$,

$(8t+8i+2, 20t-4i+20, 28t+4i+22)$, $1 \leq i \leq t+1$

and $(8t+8i+6, 20t-4i+14, 28t+4i+20)$, $1 \leq i \leq t$.

It can be verified that these prescriptions can be followed for all positive integer values of s, i.e., for $r \geqslant 2$, $n \geqslant 24$, provided that the triple indicated is not included when $s = 1$ $(t = 0)$, and provided sets are taken to be empty when the appropriate index set $\{i\}$ is empty. Solutions for the missing cases $r = 0$ and 1 $(n = 0,3,12$ and $15)$ are given below.

The numbers of solutions for the first few values of n are given in the following table, the last row being the appropriate cumulative totals of the preceding one, which gives the numbers of only those solutions in which n occurs (as a value of z).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2848	0	0	0	1	1	2	2	3	7	15	12	30	8	32	162	21
2849	0	0	0	1	2	4	6	3	10	25	12	42	8	40	202	21

The solutions for $n = 0$ and 3 are the empty set and the unique triple (1,2,3). The eight solutions for $n = 12$ are

2 4 6	1 5 6	1 5 6	2 5 7	3 4 7	1 6 7	2 6 8	3 5 8
1 9 10	2 8 10	3 7 10	3 6 9	1 8 9	4 5 9	4 5 9	2 7 9
3 8 11	4 7 11	2 9 11	1 10 11	5 6 11	3 8 11	3 7 10	4 6 10
5 7 12	3 9 12	4 8 12	4 8 12	2 10 12	2 10 12	1 11 12	1 11 12

and the twenty-one solutions for $n = 15$ are

2 4 6	1 5 6	1 5 6	1 6 7	2 5 7	1 6 7	3 4 7
1 11 12	3 9 12	2 10 12	2 9 11	1 10 11	3 8 11	2 9 11
3 10 13	2 11 13	4 9 13	5 8 13	4 9 13	4 9 13	1 12 13
5 9 14	4 10 14	3 11 14	4 10 14	6 8 14	2 12 14	6 8 14
7 8 15	7 8 15	7 8 15	3 12 15	3 12 15	5 10 15	5 10 15

2 5 7	3 4 7	1 7 8	2 6 8	3 5 8	1 7 8	2 6 8
3 8 11	1 10 11	4 6 10	3 7 10	1 9 10	5 6 11	4 7 11
1 12 13	5 8 13	2 11 13	1 12 13	6 7 13	3 9 12	3 9 12
4 10 14	2 12 14	5 9 14	5 9 14	2 12 14	4 10 14	1 13 14
6 9 15	6 9 15	3 12 15	4 11 15	4 11 15	2 13 15	5 10 15

3 5 8	3 6 9	4 5 9	1 8 9	2 7 9	3 6 9	4 5 9
4 7 11	2 8 10	3 7 10	4 6 10	5 6 11	4 7 11	3 8 11
2 10 12	5 7 12	1 11 12	5 7 12	4 8 12	2 10 12	2 10 12
1 13 14	1 13 14	6 8 14	3 11 14	3 10 13	5 8 13	6 7 13
6 9 15	4 11 15	2 13 15	2 13 15	1 14 15	1 14 15	1 14 15

R. B. Eggleton has suggested generalizing the problem to $ax + by = cz$ where $(a,b,c) = 1$. He has obtained asymptotic bounds for the number of solutions in the case $x+y = 2z$ (which has the simple solution $(3i-2,\ 3i,\ 3i-1)$, $1 \leq k \leq n$, $n = 3k$) and we hope to publish a complete solution elsewhere, at least in the case $a = b$.

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