Exact formulas for $2 \times n$ arrays of dumbbells

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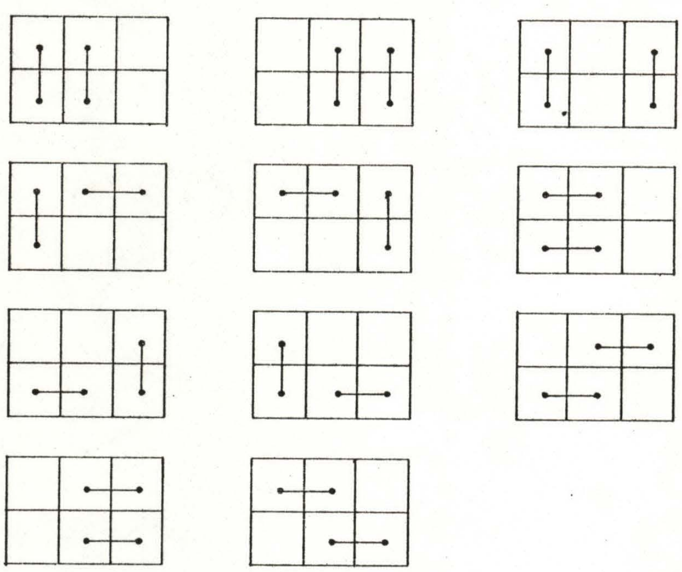
Several exact results are given for the problem of enumerating arrangements of q indistinguishable dumbbells on a $2 \times n$ array of compartments.

1. INTRODUCTION

McQuistan and Lichtman[1] have investigated the following dimer problem which has bearing on several areas of physics. Consider a $2 \times n$ rectangular array of compartments (a lattice space) and q dumbbell-shaped objects, •——•. Let $A(q, n)$ be the number of ways in which the q dumbbells may be placed in the array such that the two ends of each dumbbell are in two horizontally or vertically adjacent compartments and no two dumbbells have ends which share a compartment. For example example if $q = 2$ and $n = 3$, the possibilities are:



In this case $A(2, 3) = 11$. The following recurrence is known[1]:

$$A(q, n) = A(q, n-1) + 2A(q-1, n-1) + A(q-1, n-2) - A(q-3, n-3). \quad (1)$$

Clearly $A(q, n) = 0$ if $q > n$ so the array of numbers $A(q, n)$ is triangular, part of which is given by the following table:

n \ q	0	1	2	3	4	5
0	1					
1	1	1				
2	1	4	2			
3	1	7	11	3		
4	1	10	29	26	5	
5	1	13	56	94	56	8

McQuistan and Lichtman remark that exact solutions for problems of this sort (2 or more rows and/or 2 or more dimensions) have been obtained for only very special cases,[2,3] i.e., a 2-dimensional array completely covered with dumbbells. In many other investigations, approximation methods have been utilized.

In this paper we obtain some explicit formulas for $A(q, n)$, some generating functions, another representation of the problem, and another recurrence. Also we shall see how $A(q, n)$ is related to other well-known functions.

2. THE GENERATING FUNCTION AND ITS COEFFICIENTS

Put

$$f_n(x) = \sum_{q=0}^{n} A(q, n) x^q.$$

Then by (1) and a little manipulation,

$$f_{n+3}(x) = (2x+1) f_{n+2}(x) + x f_{n+1}(x) - x^3 f_n(x) \quad (n \geq 0). \quad (2)$$

For example,

$$f_0(x) = 1,$$
$$f_1(x) = 1 + x,$$
$$f_2(x) = 1 + 4x + 2x^2$$
$$f_3(x) = 1 + 7x + 11x^2 + 3x^3,$$

which are verified by the above table. Next put

$$G(x, y) = \sum_{n=0}^{\infty} f_n(x) y^n. \quad (3)$$

Using (2) and the first 3 equations under (2) as initial conditions, we find

$$G(x, y) = (1 - xy)/(1 - 2xy - y - xy^2 + x^3y^3). \quad (4)$$

Thus, (4) gives the ordinary bivariate generating function for $A(q, n)$. Incidentally, with $y = 1$, (4) becomes (15) of the McQuistan–Lichtman paper.[1] Clearly the usual methods of expanding (4) yield fairly complicated formulas. Of these, one of the more compact is, by the multinomial theorem,

$$A(q, n) = \sum_{i=0}^{1} \sum_{\substack{b+c+3d=q+i \\ a+2b+c+3d=n+i}} (-1)^{d+i} 2^c \binom{a+b+c+d}{a, b, c, d},$$

where $\binom{a+b+c+d}{a,b,c,d}$ is a multinomial coefficient.

Put

$$A = 1 - y,$$
$$B = -2y - y^2,$$
$$C = y^3,$$

so that, by (4),

$$G(x, y) = \frac{1 - xy}{1 - x} \left(1 + \frac{B}{A}x + \frac{C}{A}x^3\right)^{-1}. \quad (5)$$

Since A, B, and C are functions of y only, we may expand the right side of (5) to obtain $G(x,y)$ as a power series in x whose coefficients involve y [compare with (3)]. This may be expressed by

$$G(x,y)=\frac{1-xy}{1-y}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{B}{A}\right)^{k-j}\left(-\frac{C}{A}\right)^{j}x^{k+2j}. \qquad (6)$$

If we carry through with the rather tedious details of expanding the right side of (6) in terms of y, thus getting a power series of x and y, we find that the coefficient of $x^q y^n$ is another (as to be expected) complicated formula. But this time we observe some interesting results. The coefficient is

$$A(q,n)=\sum_{k=0}^{q}\sum_{j=0}^{[q/3]}(-1)^j 2^{q-3j-k}\binom{q-2j}{j}\binom{q-3j}{k}\binom{n-2j-k}{q-2j}$$
$$-\sum_{k=0}^{q-1}\sum_{j=0}^{[(q-1)/3]}(-1)^j 2^{q-3j-k-1}\binom{q-2j-1}{j}\binom{q-3j-1}{k}\binom{n-2j-k-1}{q-2j-1},$$

where $[x]$ is the largest integer less than or equal to x. Thus, we may view $A(q,n)$ as a polynomial in n of degree q. The coefficient of n^q appears in the first double sum when $j=0$. In this case ($j=0$) the first double sum becomes

$$\sum_{k=0}^{q}2^{q-k}\binom{q}{k}\frac{1}{q!}(n-k)_q.$$

But the coefficient of n^q in $(n-k)_q$ is 1 and

$$\sum_{k=0}^{q}2^{q-k}\binom{q}{k}\frac{1}{q!}=\frac{3^q}{q!}.$$

Therefore, $A(q,n)$ is a polynomial in n as follows.

$$A(q,n)=\frac{3^q}{q!}n^q+C_1n^{q-1}+\cdots+C_{q-1}n+C_q, \qquad (7)$$

where the C's depend on q only. If we put $\Delta A(q,n)=A(q, n+1)-A(q,n)$, then (7) implies the recurrence

$$\Delta^q A(q,n)=3^q.$$

3. ANOTHER FORMULA AND ASSOCIATED FUNCTIONS

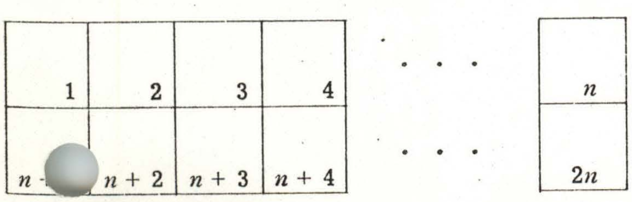
The occupation of $2\times n$ arrays with dumbbells may be expressed in terms of an occupancy problem with restricted positions. Consider 3 sets of cells labeled as follows:

$$\underbrace{\underline{\quad}\ \ \underline{\quad}\ \cdots\ \underline{\quad}}_{\text{I}}\quad \underbrace{\underline{\quad}\ \ \underline{\quad}\ \cdots\ \underline{\quad}}_{\text{II}}$$

I: $1, n+1$	$2, n+2$	$\cdots$	$n, 2n$	II: $1,2$	$2,3$	$\cdots$	$n-1, n$

III: $n+1, n+2$	$n+2, n+3$	$\cdots$	$2n-1, 2n$	(8)

Let the first n cells, I, represent the n vertical pairs of compartments of the $2\times n$ array, let the next set, II, of $n-1$ cells represent the $n-1$ horizontally adjacent pairs of compartments in the first row and let the final $n-1$ cells, III, be the horizontally adjacent pairs of compartments of the second row. Thus, (8) is equivalent to



Each of the cells of (8) has 2 "labeling" numbers. Therefore, $A(q,n)$ is the number of ways we may distribute q like objects, one per cell, into (8) such that no 2 cells containing objects have a labeling number in common. The advent that 2 occupied cells do share a labeling number is equivalent to the ends of 2 dumbbells sharing a compartment, a situation which is forbidden.

Given that j of the q objects are in certain of the cells of I, there are u and v ($u+v=q-j$) objects that are to be distributed among the nonforbidden positions of II and III respectively. For example, if $n=11, j=3$, and the checks denote the cells of I occupied by the 3 objects, then the set of cells I, II, and III are

	1	2	3	4	5	6	7	8	9	10	11
I		√	√				√				

gap of 1 (cell 1), gap of 3 (cells 4–6), gap of 4 (cells 8–11)

(deleting the second labeling number of each cell),

	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	9, 10	10, 11
II	x	x	x			x	x			
III	x	x	x			x	x			

(deleting the labeling numbers)

where the x's denote forbidden positions. Clearly II and III are always identical. If u objects occupy II then $q-j-u=v$ objects occupy III. We define a gap of m to be m successive unoccupied cells of I. Thus a gap of m gives rise to $m-1$ permissible cells in II and III. But of those $m-1$ permissible positions, no 2 adjacent ones may be occupied. The number of ways that we may place i objects in $m-1$ cells so that no 2 adjacent cells are occupied is

$$\binom{m-i}{i}.$$

(A well-known elementary fact asserts that the number of ways that r plus signs and s minus signs may be arranged in a row so that no 2 minus signs are adjacent is $\binom{r+1}{s}$; the above sentence is equivalent to this). If cell a_1 of I is occupied and if the next cell in I to the right of a_1 which is occupied is a_2 then the corresponding gap is a_2-a_1-1. Allowing j to range from 0 to q and accounting for all possible distribution of the j objects in I we find

$$A(q,n)=\sum_{j=0}^{q}\ \sum_{\substack{a_0<a_1<\cdots<a_j<n+1\\ u+v=q-j}} f(j,u)f(j,v), \qquad (9)$$

where $a_0=0$ (the other a's are indices) and where

$$f(j,u) = f(u; a_0, a_1, \ldots, a_j) = \sum_{i_1+\cdots+i_{j+1}=u} \binom{a_1-a_0-1-i_1}{i_1}\binom{a_2-a_1-1-i_2}{i_2}\cdots \binom{a_j-a_{j-1}-1-i_j}{i_j}\binom{n-a_j-i_{j+1}}{i_{j+1}}. \quad (10)$$

Thus (9) provides $A(q,n)$ with a more harmonious formula than do some of the earlier equations.

Further insight into the intricate nature of $A(q,n)$ may be made through a study of $f(j,u)$. Actually, some properties of $f(j,u)$ are well known. Putting

$$b_i = \begin{cases} a_i - a_{i-1} - 1 & (i = 1, 2, \ldots, j) \\ n - a_j & (i = j+1) \end{cases}$$

it suffices to define and examine

$$g(u; b_1, \ldots, b_r) = \sum_{i_1+\cdots+i_r=u} \binom{b_1-i_1}{i_1} \cdots \binom{b_r-i_r}{i_r}.$$

Then

$$\sum_{u=0}^{\infty} g(u; b_1, \ldots, b_r) z^u = \prod_{k=1}^{r} \sum_{i_k=0}^{b_k} \binom{b_k-i_k}{i_k} z^{i_k}. \quad (11)$$

But the function

$$u_b(z) = \sum_{i=0}^{b} \binom{b-i}{i} z^i \quad (12)$$

is familiar. The numbers $u_b(1)$ are the Fibonacci numbers. The polynomial $u_b(z)$ has been extensively studied by many investigators.[4] Two expressions for $u_b(z)$ are

$$u_b(z) = (-1)^b x^{b/2} U_b(i/2\sqrt{x}), \quad (*)$$

where $i = \sqrt{-1}$ and $U_b(z) = \sin(b+1)\theta/\sin\theta (z = \cos\theta)$; $U_b(z)$ is a Chebyshev polynomial.

$$u_b(z) = 2^{-b-1}\alpha^{-1}[(1+\alpha)^{b+1} - (1-\alpha)^{b+1}],$$
$$\alpha = (1+4x)^{1/2}. \quad (**)$$

[Compare (**) with the Binet form of the Fibonacci numbers.]

Using (11) and (12) it is easily seen that

$$\sum_{b_1,\ldots,b_r=0}^{\infty} \sum_{n=0}^{\infty} g(u; b_1, \ldots, b_r) z^u y_1^{b_1} \cdots y_r^{b_r} = \prod_{i=1}^{r} (1 - y_i - z y_i^2)^{-1}.$$

In a subsequent paper we shall show how the ideas presented in the latter part of this paper may be utilized and extended to enumerate arrangements of q dimers on an $m \times n$ lattice where it is not necessary to assume that the dimers are numerous enough to completely cover the lattice.

[1] R. B. McQuistan and S. J. Lichtman, J. Math. Phys. 11, 3095 (1970).
[2] M. E. Fisher, Phys. Rev. 124, 1664 (1961).
[3] P. W. Kasteleyn, Physica 27, 1209 (1961).
[4] J. Riordan, *Combinatorial Identities* (Wiley, New York, 1968), pp. 75, 76, 242.