Scan 3105

Lang & Trotter

many seqs

Segs!

JRAM 255 (1972) 112–134

5483 39921 39922 39923

2945 2946 2947 2948 ~~3133~~ ~~3134~~ ~~3135~~ → 3105

Continued fractions for some algebraic numbers

By *Serge Lang* and *Hale Trotter* at Princeton

The following tables at the end of the paper contain the continued fractions and some related information for a few algebraic numbers, viz:

$$\sqrt[3]{2},\ \sqrt[3]{3},\ \sqrt[3]{4},\ \sqrt[3]{5},\ \sqrt[3]{7},\ \alpha_1,\ \alpha_2,\ \alpha_3,$$

where

$$\alpha_1 = 2\cos\frac{2\pi}{7} \text{ is a root of } x^3 + x^2 - 2x - 1,$$

$$\alpha_2 \text{ is a root of } x^5 - x - 1,$$

$$\alpha_3 = \sqrt[3]{2} + \sqrt{3} \text{ is a root of } x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23.$$

The first numbers were chosen as cubic irrationalities, α_1 as being totally real, and we picked α_2, α_3 at random to be non cubic. The original motivation was to see if computations for algebraic numbers would be in line with some conjectures made in [3] and [4], and to observe whatever else might come up.

1. Table I

In each case, this table is to be read horizontally, and gives the first thousand terms in the continued fraction. For instance, for $\sqrt[3]{2}$, the continued fraction begins

$$[1, 3, 1, 5, 1, 1, 4, 1, 1, 8, \ldots].$$

The last five terms among these first thousand are $[\ldots, 2, 1, 1, 1, 1]$. Thus the continued fraction $[a_1, a_2, \ldots]$ is laid out in rows of twenty integers, and there are fifty such rows. The position of any a_n is therefore easily determined.

For clarity, and reasons of space distribution, the table is filled only with the 2-digit integers which arose. Whenever a larger integer occurred, it is indicated by the symbols *a, *b, *c, ..., and these are then listed separately at the bottom. For instance, for $\sqrt[3]{2}$, we have

$$*a = 534, \qquad *b = 121, \qquad \ldots, \qquad *k = 4941, \qquad \ldots$$

2. Table II

This table gives the frequency count for the digits appearing in the continued fraction. According to a theorem of Kuzmin [2], for almost all numbers α, the probability that the n-th integer a_n in the continued fraction for x is equal to a positive integer k is given by

$$\log_2 \frac{(k+1)^2}{k(k+2)} .$$

For $k = 1$, and almost all numbers, this means that the probability for $a_n = 1$ is approximately .41. In each case of the computations, we find that the numbers behave very closely to this generic expectation, thus confirming that in most respects, the continued fractions for algebraic numbers of degree > 2 should be essentially like those of almost all numbers. For instance, among the first thousand a_n for $\sqrt[3]{2}$, $\sqrt[13]{4}$, and α_3 we find that 1 occurs respectively 422, 412, and 418 times. The biggest divergence from the Kuzmin number is for $\sqrt[3]{5}$, and even then 433 is still in line with the expected asymptotic estimate. The Kuzmin probability that 2 occurs is approximately .17, and again in this case we find that 2 occurs 165 times, resp. 159 times, resp. 168 times for $\sqrt[3]{2}$, $\sqrt[3]{3}$, α_1 respectively. One should also note that this type of regularity is exhibited throughout, among the first thousand terms.

3. The occurrence of large integers

Aside from what appear to be routine low numbers, there occur larger numbers which seem to be of two kinds: Some which are only somewhat larger than the ordinary ones, and some which appear to be exceptionally large. For instance, for $\sqrt[3]{2}$, we meet

$$a_{572} = 7\,451 \text{ and } a_{620} = 4\,941.$$

For $\sqrt[3]{5}$, we meet

$$a_{19} = 3\,052, \qquad a_{691} = 13\,977, \qquad a_{813} = 49\,968.$$

The occurrence of large numbers in the continued fractions of certain cubic irrationalities has already been observed by Brillhart, see Churchhouse and Muir [1]. A theoretical explanation for some of them has been proposed by Stark [6], but the following problems remain: Determine whether there is a basic theoretical distinction between what seem to be only medium large numbers, and very large ones. More importantly, determine whether exceptionally large integers will continue to occur throughout the continued fraction, or whether they will stop from occurring. The explanation given by Stark depends on some class numbers being equal to 1, and thus would account for only a finite number of them. In general, the appearance of such large integers may depend on the arithmetic properties of the field obtained from the square root of the discriminant, e. g. its class number. The tables seem to indicate that they stop.

To discuss the statistical significance of exceptionally large values of a_n occurring near the beginning of the sequence of partial quotients, we need an estimate of the probability $q_{N,K}$ that the first N partial quotients of a "random" number are all less than a given integer K. It is perhaps most natural to consider a random number as distributed uniformly on (0, 1), but in this context the distribution given by

$$Pr\{X \leq c\} = \log_2 (1 + c), \quad \text{for } c \in (0, 1),$$

is more appropriate, because if X has this distribution, then the distribution of the partial quotient $a_n(X)$ is independent of n. In fact,

$$Pr\{a_n(X) < K\} = \gamma_K,$$

where

$$\gamma_K = Pr\{X^{-1} < K\} = 1 - \log_2\left(1 + \frac{1}{K}\right),$$

which is the Kuzmin theorem already alluded to.

To see that this is so, observe that $a_n(X) = [X_n^{-1}]$, where $X_1 = X$, and for $n > 0$,

$$X_{n+1} = X_n^{-1} - [X_n^{-1}].$$

As usual, $[x]$ is the largest integer $\leqq x$. It is then an exercise in calculus to show that if f_n is a density function for X_n, so that

$$Pr\{X_n \leqq c\} = \int_0^c f_n(x)\,dx,$$

then $f_{n+1} = Tf_n$ is a density function for X_{n+1}, where T is the linear operator on $L^1(0, 1)$ defined by

$$(Tf)(x) = \sum_{k=1}^{\infty} (x+k)^{-2} f((x+k)^{-1}).$$

It follows that $f_{n+1} = T^n f_1$. It is easy to verify that the function

$$\frac{1}{(\log 2)(1+x)}$$

is a density function and is invariant under T, so that if X has this density, in which case, by integration,

$$Pr\{X \leqq c\} = \log_2(1+c),$$

all the X_n have the same distribution. In fact Kuzmin's theorem states that if f is any smooth probability density, then

$$\lim_{n\to\infty} T^n f(x) = \frac{1}{(\log 2)(1+x)}.$$

It follows that as $n \to \infty$ the distribution of $a_n(X)$ tends to the one given above if X has any smooth distribution. For a discussion of all these ideas and a proof of Kuzmin's theorem, see [2].

If the random variables $a_n(X)$ were independent, then we would have

$$q_{N,K} = \gamma_K^N.$$

This is not strictly correct, but can be expected to give a good approximation for large values of N and K. A combination of theoretical and numerical analysis indicated strongly that the relative error is bounded by $\lambda N K^{-2}$, with $\lambda < 1$, and we are confident that the approximation is entirely adequate for our purposes.

The short Table A at the end of this section shows, for each of the numbers investigated, the maximum value A of the first 1000 partial quotients, and the value M for which $a_M = A$. The third column gives

$$p_{1000,A} = 1 - q_{1000,A},$$

the probability that a "random" number would have a value as large as A among its first 1000 partial quotients. The smaller the value of p, the stronger the evidence that the number is unusual. The fourth column gives

$$p_{M,A} = 1 - q_{M,A},$$

the probability of getting a value as large as A among the first M quotients. The value is of course smaller than that of p, and its statistical meaning less clear since M is taken a posteriori to make the probability small.

The table shows that if one goes by the maximum quotient found, only $\sqrt[3]{5}$ appears highly unusual, although one might question $\sqrt[3]{4}$ and α_2. If one also takes into account the second largest quotient, then $\sqrt[3]{5}$ with $a_{691} = 13977$ appears even more unusual, and $\sqrt[3]{2}$ with $a_{620} = 4941$ perhaps comes to be of interest also.

Table A

Number	A	M	$p = 1 - q_{1000,A}$	$p_M = 1 - q_{M,A}$
$\sqrt[3]{2}$	7451	572	.18	.10
$\sqrt[3]{3}$	3502	916	.34	.31
$\sqrt[3]{4}$	14902	579	.09	.05
$\sqrt[3]{5}$	49968	813	.03	.02
$\sqrt[3]{7}$	689	611	.88	.72
α_1	904	830	.80	.73
α_2	11644	588	.12	.07
α_3	1446	54	.63	.05

4. Table III

In each case, this table begins with the columns labelled n, a_n, and q_n. The n indicates n-th position in the continued fraction. The a_n means the n-th partial quotient. The q_n means the denominator in the approximating fraction p_n/q_n (classical notation). For instance, in the case of $\sqrt[3]{2}$, we have

$$a_{36} = 534, \qquad q_{36} = 3.06 \times 10^{19}$$

$$a_{42} = 121, \qquad q_{42} = 8.95 \times 10^{22},$$

and so forth. In machine language, $E\,19$ means multiplication by 10^{19}, and $E\,486$ means multiplication by 10^{486} (the last line in Table III).

Table III includes these data for all n among the first thousand such that $a_n \geqq 50$. We picked 50 as a cutting point after looking at preliminary computations, because it included all the numbers a_n which could be labelled as somewhat large, and at the same time provided only a rather small table.

The last column r_n in Table III gives (up to three decimals) the quotient

$$\frac{q_n}{q_{n-1} \log q_{n-1}}$$

for those values of n when $a_n \geqq 50$. The reason for this quotient to be interesting are as follows. According to a theorem of Roth, if α is algebraic, there is only a finite number of integers $q > 0$, p such that

$$| q\alpha - p | < \frac{1}{q^{1+\varepsilon}}.$$

It was suggested in [3] and [4] that this theorem should be improvable by an inequality

$$| q\alpha - p | < \frac{1}{qf(q)},$$

where f is a function close to the logarithm, for instance $(\log q)^{1+\varepsilon}$, or perhaps even $\log q$ itself, up to a constant factor of course. Such a function is called a type in [4]. If some a_n is small, then q_n/q_{n-1} being approximately equal to a_n shows that the quotient

$$\frac{q_n}{q_{n-1} \log q_{n-1}}$$

is approximately like $1/\log q_{n-1}$. Thus to investigate the possibility of a type f, we look at those n for which a_n is comparatively large. Again n such that $a_n \geqq 50$ seemed to give the most information for the least amount of space used. It is even unsolved for any algebraic number of degree > 2 whether it is of bounded type, but the tables seem to fall fairly well in line with expectations, e. g. differing from the log by a function with a lower order of magnitude (above or below).

We have also programmed the same data for the first 3000 terms of the continued fractions of the cubic numbers listed. In every case, exceptionally large integers did not seem to recur, and generally speaking, the ratio r_n seems to decrease. We thought it pointless to reproduce these more extensive tables in full, but we give in Table B the portion of Table III for $n > 1000$ when $r_n > 1$, rounding off r_n to one decimal.

The tables therefore suggest that the type may in fact not be bigger than a constant times the logarithm, and may even be of an order of magnitude smaller than the logarithm. Following certain asymptotic estimates of Adams, who looked at the continued fraction of e, it was shown (cf. [4]) that the type of e is asymptotic to $\log q/\log\log q$. Thus one is beginning to be accustomed to such small types. Note that for a function essentially not bigger than the log, the series

$$\Sigma \frac{1}{qf(q)}$$

diverges, so that these cases go very slightly against the Khintchine convergence principle: If ψ is such that $\Sigma\, \psi(q)$ diverges (resp. converges), then for almost all numbers, the inequality

$$| q\alpha - p | < \psi(q)$$

has infinitely many solutions (resp. only a finite number). However, this statistical result is delicate to use for specific numbers with a type in the range of the log, because one also

Table B

	n	a_n	q_n	r_n
$\sqrt[3]{2}$	1191	12737	7.74 E 1010	5.5
	2248	2897	2.97 E 1136	1.1
$\sqrt[3]{3}$	1988	2967	3.47 E 1024	1.3
	2407	9559	1.25 E 1242	3.3
$\sqrt[3]{4}$	1974	6368	4.88 E 1010	2.7
	2248	4157	6.92 E 1146	1.6
$\sqrt[3]{5}$	1196	18905	1.47 E 600	13.8
$\sqrt[3]{7}$	None			
$2\cos\frac{2\pi}{7}$	1102	1374	6.84 E 576	1.0

knows that if α is a number such that for every function ψ (decreasing) having convergent [illegible]n, the above inequality has only a finite number of solutions, then α must be of bounded type. For all this, cf. [4].

5. Computational method

The computations were done by the following algorithm, which uses integer arithmetic only, and thus involves no rounding error.

Given a polynomial $p_n(x)$, of degree d, with positive leading coefficient and a unique positive root y_n which is simple, irrational, and greater than 1, we construct a polynomial $P_{n+1}(x)$ as follows. Let $a_n = [y_n]$ be the greatest integer such that $P_n(a_n) < 0$. Define

$$Q_n(x) = P_n(x + a_n) \text{ and } P_{n+1}(x) = -x^d Q_n(x^{-1}).$$

Then $Q_n(x)$ has exactly one root between 0 and 1, and since the roots of $P_{n+1}(x)$ are the reciprocals of the roots of $Q_n(x)$, we see that $P_{n+1}(x)$ has a unique positive root y_{n+1}. Obviously y_{n+1} is also a simple root, irrational, and greater than 1. Note that the constant term of $Q_n(x)$ is $P_n(a_n) < 0$, so that $P_{n+1}(x)$ also has a positive leading coefficient. Thus $P_{n+1}(x)$ has the properties assumed for $P_n(x)$, and starting from any $P_1(x)$ with these properties, we can define an infinite sequence $P_1(x)$, $P_2(x)$, ... with associated positive roots $y_1, y_2, \ldots$

We have

$$a_n = [y_n] \text{ and } y_{n+1} = (y_n - a_n)^{-1}.$$

This is precisely equivalent to saying that $a_1, a_2, \ldots$ is the sequence of partial quotients in the continued fraction expansion of y_1. If $P_1(x)$ has integer coefficients, then so has every $P_n(x)$, and the calculation involves only addition and multiplication of integers.

SeqN580.5 = 2945

2945

Table I

$\sqrt[3]{2}$

read across

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	3	1	5	1	1	4	1	1	8	1	14	1	10	2	1	4	12	2	3
20	2	1	3	4	1	1	2	14	3	12	1	15	3	1	4	*a	1	1	5	1
40	1	*b	1	2	2	4	10	3	2	2	41	1	1	1	3	7	2	2	9	4
60	1	3	7	6	1	1	2	2	9	3	1	1	69	4	4	5	12	1	1	5
80	15	1	4	1	1	1	1	1	89	1	22	*c	6	2	3	1	3	2	1	1
100	5	1	3	1	8	9	1	26	1	7	1	18	6	1	*d	3	13	1	1	14
120	2	2	2	1	1	4	3	2	2	1	1	9	1	6	1	38	1	2	25	1
140	4	2	44	1	22	2	12	11	1	1	49	2	6	8	2	3	2	1	3	5
160	1	1	1	3	1	2	1	2	4	1	1	3	2	1	9	4	1	4	1	2
180	1	27	1	1	5	5	1	3	2	1	2	2	3	1	4	2	2	8	4	1
200	6	1	1	1	36	9	13	9	3	6	2	5	1	1	1	2	10	21	1	1
220	1	2	1	2	6	2	1	6	19	1	1	18	1	2	1	1	1	27	1	1
240	10	3	11	38	7	1	1	1	3	1	8	1	5	1	5	4	4	4	7	2
260	1	21	1	1	5	10	3	1	72	6	9	1	3	3	2	1	4	2	1	1
280	1	1	2	1	7	8	1	2	1	8	1	8	3	1	1	3	2	1	8	1
300	1	1	1	1	6	1	4	3	4	1	1	1	4	30	39	2	1	3	8	1
320	1	2	1	3	1	9	1	4	1	2	2	1	6	2	1	1	3	1	4	1
340	2	1	1	5	1	2	10	1	5	4	1	1	4	1	2	1	1	2	12	2
360	1	8	3	2	6	1	3	10	1	2	20	1	6	1	2	*e	2	2	1	2
380	47	1	19	2	2	1	1	1	2	1	1	3	2	8	1	18	3	5	39	1
400	2	1	1	1	1	4	1	5	2	6	3	1	1	1	4	2	1	6	1	1
420	*f	1	3	1	3	1	4	5	1	2	1	13	2	2	2	1	1	1	1	7
440	2	1	7	1	3	1	1	11	1	2	2	4	2	33	3	1	1	2	6	3
460	1	1	3	6	8	3	4	84	1	1	2	1	10	2	2	20	1	3	1	7
480	13	14	1	29	1	1	5	1	7	1	1	2	1	56	1	3	2	1	13	2
500	1	2	2	2	1	1	1	1	1	1	*g	2	4	5	1	1	1	3	1	3
520	3	1	6	1	1	6	1	71	1	9	1	2	1	11	5	1	25	1	6	67
540	2	9	6	1	5	2	15	1	2	48	2	7	1	3	1	4	21	1	1	2
560	1	27	3	26	2	1	1	2	5	7	3	*h	2	29	4	3	8	17	3	8
580	2	3	1	1	1	5	*i	1	3	4	1	4	1	1	13	1	34	1	2	7
600	1	3	3	7	1	3	1	1	4	2	69	1	3	12	34	1	2	*j	1	*k
620	4	1	1	12	3	4	2	3	1	1	1	1	1	2	1	1	6	16	1	2
640	27	2	13	4	1	1	1	3	11	1	1	3	1	53	2	15	1	1	1	1
660	1	1	2	2	1	1	3	3	1	9	1	1	10	3	1	1	2	1	2	2
680	1	10	9	1	2	5	1	2	2	1	1	2	4	7	1	5	1	1	1	1
700	4	2	25	16	5	4	1	3	2	3	13	1	49	6	2	5	1	1	2	7
720	3	2	1	1	1	4	1	1	1	5	1	2	1	2	1	1	1	1	2	2
740	4	1	2	1	10	5	4	8	10	2	4	1	1	1	4	1	41	1	3	1
760	56	3	1	1	3	1	3	1	5	6	6	3	1	2	1	1	1	12	1	10
780	2	1	1	1	1	50	5	1	2	6	5	1	2	5	6	5	2	77	1	4
800	2	1	1	1	1	1	4	2	1	2	1	1	1	1	1	6	2	1	1	7
820	1	5	1	1	1	1	2	2	1	1	5	2	1	5	1	1	1	4	1	2
840	17	1	20	7	4	2	1	1	1	2	1	4	7	3	4	3	3	5	31	1
860	1	2	2	6	1	1	1	1	1	1	1	1	6	1	6	1	1	23	20	1
880	22	16	4	2	1	3	2	1	1	2	5	5	1	1	15	3	1	1	2	1
900	1	1	4	2	1	2	23	6	10	3	2	3	6	2	1	1	1	1	1	1
920	4	3	2	1	2	1	4	10	7	1	1	1	1	3	3	2	*m	1	1	11
940	2	6	1	4	1	2	2	9	3	1	1	3	22	4	1	93	1	3	1	4
960	2	1	2	3	2	1	2	11	1	1	3	1	2	1	28	23	4	11	1	9
980	1	4	3	1	6	1	2	1	12	2	6	19	1	4	4	2	1	1	1	1

a = 534 b = 121 c = 186 d = 372 e = 186 f = 220

g = 255 h = 7451 i = 113 j = 151 k = 4941 m = 108

Table II

$\sqrt[3]{2}$

FREQUENCY COUNTS

1	422	22	4	56	2
2	165	23	3	67	1
3	91	25	3	69	2
4	66	26	2	71	1
5	40	27	4	72	1
6	37	28	1	77	1
7	20	29	2	84	1
8	16	30	1	89	1
9	15	31	1	93	1
10	15	33	1	108	1
11	8	34	2	113	1
12	9	36	1	121	1
13	8	38	2	151	1
14	4	39	2	186	2
15	5	41	2	220	1
16	3	44	1	255	1
17	2	47	1	372	1
18	3	48	1	534	1
19	3	49	2	4941	1
20	4	50	1	7451	1
21	3	53	1		

Table III

$\sqrt[3]{2}$

n	a_n	q_n	r_n
36	534	3.06 E 19	13.844
42	121	8.95 E 22	2.530
73	69	4.48 E 39	0.799
89	89	3.61 E 48	0.835
92	186	1.56 E 52	1.618
115	372	6.69 E 65	2.560
269	72	1.90 E 146	0.219
376	186	4.78 E 194	0.421
421	220	4.51 E 216	0.447
468	84	1.19 E 239	0.154
494	56	1.01 E 253	0.098
511	255	3.59 E 260	0.430
528	71	3.38 E 268	0.117
540	67	1.32 E 276	0.106
572	7451	8.64 E 297	11.005
587	113	4.07 E 308	0.160
611	69	7.64 E 320	0.095
618	151	5.97 E 326	0.202
620	4941	2.97 E 330	6.568
654	53	1.50 E 347	0.068
761	56	5.13 E 395	0.063
786	50	7.54 E 406	0.054
798	77	4.14 E 414	0.082
937	108	3.81 E 475	0.099
956	93	1.37 E 486	0.084

N155.5 = 2946

Table I

$\sqrt[3]{3}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	2	3	1	4	1	5	1	1	6	2	5	8	3	3	4	2	6	4	4
20	1	3	2	3	4	1	4	9	1	8	4	3	1	3	2	6	1	6	1	3
40	1	1	1	1	12	3	1	3	1	1	4	1	6	1	5	1	2	1	3	3
60	11	8	1	*a	8	2	8	5	1	2	2	2	2	3	1	1	2	1	1	1
80	52	2	46	2	2	3	3	1	5	1	6	1	6	1	1	1	6	1	20	10
100	4	8	1	1	1	2	1	2	*b	1	2	5	6	1	4	3	1	1	*c	2
120	3	3	1	6	1	3	1	2	1	24	2	1	5	4	1	1	2	1	1	1
140	3	1	1	20	5	60	1	1	37	6	3	28	1	1	2	1	31	1	3	3
160	1	2	58	1	11	1	31	1	8	1	11	2	2	3	1	4	1	4	37	1
180	2	1	2	82	1	2	6	11	5	1	1	1	7	54	2	27	1	2	1	24
200	3	3	1	2	1	1	*d	1	14	2	89	1	2	4	1	5	2	1	2	2
220	3	2	1	5	1	4	2	1	15	2	2	3	10	2	1	1	1	1	1	1
240	9	2	67	1	1	1	9	5	2	1	3	1	60	1	3	1	1	3	1	9
260	2	18	1	3	1	1	3	3	2	1	3	1	5	4	2	1	1	3	1	6
280	2	39	1	3	4	1	2	1	1	2	1	1	4	3	1	1	4	2	3	1
300	4	1	1	1	1	1	5	*e	1	84	3	1	2	1	3	7	3	1	1	1
320	8	1	7	1	1	1	11	1	1	2	1	1	5	1	1	1	3	1	1	2
340	2	1	7	24	5	4	1	1	1	2	2	1	1	95	1	3	1	2	1	6
360	2	1	1	6	1	1	1	3	1	16	*f	6	4	1	9	4	3	*g	1	2
380	3	4	2	1	1	*h	2	6	4	2	1	1	5	2	4	1	1	2	1	7
400	6	1	2	10	*i	3	1	2	1	1	34	1	1	2	2	1	1	10	4	15
420	1	2	1	1	2	4	1	3	1	7	1	42	1	3	1	2	4	6	2	1
440	2	1	28	3	1	5	3	1	1	1	3	2	4	4	11	1	1	3	2	10
460	6	2	1	1	26	3	2	1	1	1	1	2	26	2	3	1	66	6	1	8
480	2	1	4	1	10	3	1	1	2	1	1	1	24	4	1	2	23	4	8	1
500	41	1	4	1	1	25	1	4	1	4	6	23	1	5	2	23	1	4	3	1
520	1	5	16	1	8	2	1	11	2	2	1	1	10	3	58	4	1	34	1	1
540	1	19	1	1	1	1	1	1	1	1	1	1	7	3	3	1	11	1	16	1
560	6	5	19	7	2	4	2	2	7	4	1	3	1	1	1	3	1	1	3	*j
580	8	1	1	10	6	2	8	23	5	2	1	17	2	2	15	1	1	1	20	1
600	3	6	1	1	3	1	1	1	4	2	26	1	2	12	2	8	1	15	2	3
620	54	3	2	22	1	1	3	1	2	1	92	1	1	1	4	1	3	4	4	1
640	1	1	1	1	12	2	1	18	1	5	9	1	1	5	3	1	1	1	9	2
660	1	4	1	2	1	12	1	1	15	1	1	1	3	1	6	2	2	2	12	21
680	1	3	1	15	3	4	4	6	1	10	1	1	1	1	1	5	2	4	25	2
700	3	1	2	2	2	3	9	71	14	2	1	1	1	2	4	1	1	2	*k	1
720	1	*m	1	1	1	1	1	6	3	1	*n	5	12	2	2	*p	4	9	1	1
740	1	3	1	1	3	7	1	1	5	1	*q	1	34	1	12	2	1	1	2	1
760	69	1	2	3	2	*r	1	1	4	1	2	1	1	10	1	5	3	1	1	1
780	1	4	2	1	2	3	1	3	10	21	1	1	1	1	2	1	13	1	10	2
800	1	4	21	39	1	19	1	1	5	1	38	1	2	1	28	56	5	3	1	1
820	2	11	2	2	1	1	1	1	5	2	11	5	3	1	2	1	6	4	1	4
840	1	3	18	1	1	1	1	6	3	1	5	1	2	1	1	1	54	1	16	2
860	1	1	20	1	1	1	9	2	2	21	1	12	2	2	1	1	1	1	4	1
880	1	5	2	1	1	1	2	3	4	1	13	2	1	8	1	*s	2	1	60	1
900	1	6	10	8	4	1	3	1	2	5	1	3	16	20	5	*t	1	1	7	8
920	1	4	1	1	2	7	4	1	33	1	1	1	12	1	11	3	2	1	11	2
940	41	5	1	1	2	17	1	1	1	2	7	2	2	2	2	1	1	1	1	1
960	15	2	28	2	3	1	3	13	4	1	1	1	3	72	1	13	8	1	2	1
980	2	2	3	1	6	1	1	3	1	*u	1	2	4	4	3	15	7	2	39	2

a = 139, b = 249, c = 612, d = 220, e = 123, f = 131
g = 196, h = 729, i = 164, j = 396, k = 343, m = 137
n = 139, p = 268, q = 247, r = 1232, s = 116, t = 3502
u = 164

Table II

$\sqrt[3]{3}$

FREQUENCY COUNTS

1	425	13	4	25	2	46	1	84	1	247	1
2	159	14	2	26	3	52	1	89	1	249	1
3	97	15	8	27	1	54	3	92	1	268	1
4	64	16	5	28	4	56	1	95	1	343	1
5	37	17	2	31	2	58	2	116	1	396	1
6	32	18	3	33	1	60	3	123	1	612	1
7	14	19	3	34	3	66	1	131	1	729	1
8	18	20	5	37	2	67	1	137	1	1232	1
9	10	21	4	38	1	69	1	139	2	3502	1
10	13	22	1	39	3	71	1	164	2		
11	12	23	4	41	2	72	1	196	1		
12	9	24	4	42	1	82	1	220	1		

Table III

$\sqrt[3]{3}$

n	a_n	q_n	r_n
64	139	6.85 E 30	2.118
81	52	9.42 E 38	0.614
109	249	3.42 E 54	2.077
119	612	8.70 E 60	4.575
146	60	7.46 E 73	0.363
163	58	1.94 E 84	0.307
184	82	2.18 E 96	0.379
194	54	3.33 E 102	0.233
207	220	3.57 E 110	0.885
211	89	9.94 E 113	0.347
243	67	3.00 E 128	0.231
253	60	4.08 E 133	0.200
308	123	1.53 E 157	0.345
310	84	1.31 E 159	0.235
354	95	5.96 E 177	0.236
371	131	9.09 E 185	0.310
378	196	7.22 E 191	0.450
386	729	1.24 E 197	1.631
405	164	4.22 E 207	0.347
477	66	2.34 E 242	0.121
535	58	2.72 E 274	0.093
580	396	9.29 E 296	0.585
621	54	5.56 E 320	0.074
631	92	2.84 E 326	0.124
708	71	2.10 E 363	0.085
719	343	5.43 E 369	0.406
722	137	1.49 E 372	0.161
731	139	4.61 E 376	0.162
736	268	3.91 E 381	0.307
751	247	3.47 E 389	0.278
761	69	2.10 E 395	0.077
766	1232	6.02 E 399	1.349
816	56	2.01 E 425	0.057
857	54	6.13 E 443	0.054
896	116	1.77 E 462	0.110
899	60	3.23 E 464	0.057
916	3502	3.38 E 477	3.209
974	72	1.22 E 507	0.062
990	164	1.91 E 515	0.139

N74.5 = 2947

Table I

$\sqrt[3]{4}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	1	1	2	2	1	3	2	3	1	3	1	30	1	4	1	2	9	6	4
20	1	1	2	7	2	3	2	1	6	1	1	1	25	1	7	7	1	1	1	1
40	*a	1	3	2	1	3	60	1	5	1	8	5	6	1	4	20	1	4	1	1
60	14	1	4	4	1	1	1	1	7	3	1	1	2	1	3	1	4	4	1	1
80	1	3	1	34	8	2	10	6	3	1	2	31	1	1	1	4	3	44	1	45
100	93	12	1	7	1	1	5	12	1	1	2	4	19	1	12	1	16	1	8	1
120	1	2	1	*b	1	1	1	6	3	1	6	1	2	2	2	3	2	6	1	5
140	20	1	2	1	78	2	1	12	2	2	4	22	2	11	4	6	23	99	1	12
160	4	4	1	1	2	7	2	1	4	1	1	2	1	2	1	9	7	1	2	4
180	1	1	1	1	10	2	56	11	2	1	7	1	2	1	4	1	1	9	1	4
200	4	9	1	2	1	4	1	17	1	1	4	26	4	1	1	1	12	1	11	3
220	1	20	10	1	4	2	5	3	5	1	2	1	1	9	3	1	8	1	6	3
240	13	1	3	5	6	5	1	1	18	1	1	3	3	8	1	3	1	12	1	2
260	8	2	8	3	1	2	44	11	5	7	1	35	1	1	2	1	1	4	2	1
280	1	1	1	5	1	1	1	2	4	6	1	3	17	2	18	1	3	1	1	1
300	3	1	1	5	1	3	1	4	3	3	2	2	6	2	3	9	15	78	1	2
320	1	1	1	3	1	3	1	2	1	1	20	1	1	1	5	1	2	3	5	8
340	1	1	1	6	12	2	1	4	1	11	2	3	1	1	1	6	5	6	5	1
360	3	1	1	1	4	3	2	1	1	1	4	1	5	10	2	3	2	1	*c	1
380	5	2	1	23	2	9	1	2	2	4	2	3	1	1	2	1	3	1	37	1
400	1	1	2	79	2	4	10	1	2	4	3	7	3	2	5	1	2	1	3	*d
420	2	1	1	8	1	1	1	1	2	2	1	2	6	1	2	2	2	4	15	1
440	2	3	1	8	24	2	2	1	1	1	2	1	16	7	5	3	7	7	3	16
460	1	1	1	1	1	1	41	1	3	1	2	5	4	1	41	1	1	2	3	1
480	1	6	29	1	14	3	1	2	2	3	1	3	1	2	28	2	1	1	2	27
500	1	2	1	4	1	3	4	*e	1	8	2	1	4	1	1	2	1	1	1	1
520	2	3	3	1	2	1	*f	1	4	2	1	2	5	1	1	2	2	12	1	13
540	33	1	2	1	4	13	1	2	4	7	1	5	24	4	3	1	8	1	1	1
560	1	10	3	2	55	1	1	1	12	1	2	3	1	10	3	1	1	1	*g	1
580	58	2	6	4	34	1	1	1	3	1	2	1	1	3	11	56	1	7	2	2
600	2	3	1	6	2	17	2	1	15	1	1	6	3	1	8	9	1	*h	1	1
620	24	17	2	1	*i	1	*j	9	25	1	1	1	1	1	2	1	1	3	4	2
640	3	3	33	2	1	13	4	6	1	1	1	1	4	1	1	23	8	1	26	4
660	7	1	4	4	2	2	3	1	1	1	1	2	4	1	3	5	7	6	2	2
680	21	4	1	5	2	1	5	1	3	2	1	1	1	1	3	2	2	1	4	9
700	1	50	8	10	2	2	1	1	2	1	1	27	1	24	12	1	11	5	3	1
720	1	1	5	3	2	3	12	2	6	4	2	2	1	1	1	6	1	4	1	1
740	2	8	4	20	1	9	3	2	2	20	1	8	1	27	1	1	1	3	1	1
760	2	1	1	11	3	12	1	1	6	3	6	2	5	5	4	1	24	1	1	2
780	2	1	12	2	1	5	2	1	1	2	1	1	2	4	38	1	9	1	3	4
800	1	1	1	2	6	4	13	1	3	1	3	2	2	1	4	5	1	3	1	2
820	5	1	2	3	10	2	1	8	2	10	14	2	5	3	1	2	2	14	1	1
840	1	1	1	1	1	6	2	1	1	15	3	2	2	1	2	1	4	4	3	3
860	2	3	3	1	11	41	1	10	1	1	7	1	1	1	1	2	7	1	3	2
880	1	2	11	31	1	1	3	1	3	9	1	2	1	46	3	20	1	1	2	1
900	1	12	1	3	4	9	1	1	2	6	1	1	1	1	4	1	1	3	3	3
920	1	1	1	1	4	54	3	1	5	4	3	2	2	2	1	4	4	1	1	1
940	3	1	1	44	2	2	46	1	8	1	1	1	2	5	1	1	2	5	5	1
960	3	1	1	6	1	13	1	1	11	8	5	1	20	1	1	1	1	1	2	3
980	2	1	2	6	4	3	39	1	1	1	1	1	1	2	4	3	5	6	2	10

a = 266   b = 745   c = 372   d = 110   e = 511   f = 144
g = 14902   h = 139   i = 303   j = 2470

Table II

$\sqrt[3]{4}$

FREQUENCY COUNTS

1	412	22	1	54	1
2	164	23	3	55	1
3	100	24	5	56	2
4	69	25	2	58	1
5	39	26	2	60	1
6	32	27	3	78	2
7	19	28	1	79	1
8	20	29	1	93	1
9	14	30	1	99	1
10	12	31	2	110	1
11	11	33	2	139	1
12	15	34	2	144	1
13	6	35	1	266	1
14	4	37	1	303	1
15	4	38	1	372	1
16	3	39	1	511	1
17	4	41	3	745	1
18	2	44	3	2470	1
19	1	45	1	14902	1
20	8	46	2		
21	1	50	1		

Table III

$\sqrt[3]{4}$

n	a_n	q_n	r_n
41	266	1.93 E 19	6.868
47	60	5.59 E 22	1.248
101	93	9.85 E 51	0.808
124	745	8.43 E 65	5.136
145	78	2.00 E 76	0.460
158	99	5.29 E 86	0.508
187	56	1.53 E 100	0.249
318	78	2.14 E 167	0.205
379	372	6.02 E 194	0.842
404	79	1.26 E 207	0.168
420	110	2.84 E 216	0.223
508	511	4.53 E 260	0.861
527	144	4.29 E 268	0.236
565	55	1.14 E 289	0.084
579	14902	1.09 E 298	22.025
581	58	6.43 E 299	0.086
596	56	2.54 E 308	0.079
618	139	9.69 E 320	0.191
625	303	7.55 E 326	0.407
627	2470	1.87 E 330	3.283
702	50	2.38 E 367	0.060
926	54	2.40 E 475	0.050

2948

Table I

$\sqrt[3]{5}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	4	3	3	1	5	1	1	4	10	17	1	14	1	1	*a	1
20	1	1	1	1	1	2	2	1	3	2	1	13	5	1	1	1	13	2	41	1
40	4	12	1	5	2	7	1	1	3	33	2	1	1	1	1	1	1	3	2	2
60	1	15	12	8	10	48	1	2	1	1	3	4	1	*b	1	13	2	4	1	1
80	49	3	10	1	8	1	1	1	1	4	1	60	7	2	2	2	3	3	2	1
100	3	2	1	61	1	10	2	8	3	4	4	2	33	2	1	1	1	7	3	4
120	2	1	1	1	42	1	6	1	3	1	2	1	3	2	1	1	1	1	1	1
140	2	1	2	3	2	2	1	3	1	1	3	12	2	45	3	7	31	1	6	5
160	2	1	1	3	1	1	5	2	3	1	1	4	1	1	1	4	1	6	1	1
180	2	3	2	1	8	1	7	1	35	2	8	6	1	4	2	1	3	1	2	1
200	12	1	7	20	1	2	42	1	1	2	5	1	3	1	81	2	21	1	5	1
220	1	4	8	9	1	52	1	2	1	2	2	5	1	13	2	3	1	1	1	9
240	2	1	3	13	2	17	2	2	10	1	11	1	2	1	2	9	2	1	1	1
260	2	11	1	5	2	1	9	1	3	1	17	1	6	1	1	1	51	2	*c	2
280	8	1	1	1	1	1	25	1	5	6	3	1	2	2	3	4	1	4	9	4
300	1	1	2	1	1	1	1	1	1	1	2	4	1	2	5	1	2	11	1	1
320	1	1	1	2	1	3	3	2	1	21	1	2	1	2	1	1	38	1	4	2
340	1	5	19	4	1	1	1	2	1	2	1	1	2	4	1	2	1	3	1	1
360	2	1	1	1	4	48	5	1	6	1	1	*d	1	3	*e	23	1	51	12	2
380	35	1	1	2	3	2	1	59	1	1	2	2	3	1	1	2	3	5	2	1
400	2	5	2	1	3	*f	1	5	2	1	4	3	2	5	61	2	1	5	1	2
420	1	5	2	1	2	1	3	6	3	2	9	8	1	1	8	13	1	1	2	1
440	5	1	1	21	9	1	2	1	1	3	1	1	2	2	4	1	*g	1	1	2
460	1	6	1	50	3	2	1	4	1	1	20	5	4	2	1	1	1	1	1	1
480	7	9	10	2	9	1	*h	1	12	2	5	2	2	1	*i	1	6	1	5	1
500	9	1	4	1	23	1	3	15	3	1	2	2	2	10	2	1	25	3	2	55
520	1	1	2	1	3	2	1	7	2	2	5	1	18	3	1	2	8	3	1	3
540	1	1	5	5	1	1	1	1	4	1	1	21	1	8	1	3	2	1	5	1
560	4	3	1	2	1	13	1	1	1	2	2	1	13	2	1	5	72	1	1	4
580	1	2	1	1	1	10	2	1	3	1	3	3	1	4	2	2	1	6	6	1
600	1	1	1	1	1	29	2	2	3	1	1	3	2	1	1	1	1	2	6	1
620	1	3	1	28	20	3	1	6	2	2	1	6	1	29	1	2	15	5	3	3
640	1	5	1	1	2	1	2	2	5	3	1	1	3	2	6	7	2	1	1	4
660	4	3	1	1	3	1	3	1	3	1	37	1	8	15	1	2	4	1	10	9
680	1	2	3	1	4	1	3	7	1	1	*j	1	1	1	3	1	4	1	1	2
700	5	9	3	8	2	2	5	2	6	4	1	3	7	2	2	1	4	1	8	1
720	2	1	5	1	1	1	2	1	1	8	4	4	2	13	3	1	7	1	45	2
740	2	1	5	3	3	2	1	2	5	13	1	2	1	7	10	1	3	1	1	1
760	1	75	7	1	63	1	2	1	3	5	1	1	5	1	1	1	3	1	1	3
780	1	1	5	2	2	1	1	1	1	16	2	1	3	1	1	3	1	1	1	9
800	1	37	6	1	1	1	40	2	*k	1	5	1	*m	7	4	1	1	1	7	2
820	1	2	2	2	2	4	1	1	2	2	4	1	5	2	8	2	3	5	1	5
840	3	7	5	1	1	1	53	4	12	1	3	2	7	8	2	9	2	47	1	1
860	2	6	3	1	37	2	1	1	1	1	*n	13	1	6	2	1	2	3	2	1
880	1	3	4	1	1	4	1	2	1	1	3	1	1	2	1	2	2	3	1	1
900	3	1	6	1	7	1	1	1	4	2	1	4	1	1	2	23	1	1	1	1
920	5	1	3	1	10	16	*p	1	6	1	9	1	1	2	1	13	1	8	2	1
940	3	8	23	4	2	1	2	9	1	11	1	1	3	1	2	4	1	6	2	1
960	1	13	14	1	12	2	1	6	1	53	1	3	3	5	1	1	1	1	2	1
980	1	6	1	1	25	2	1	1	1	34	4	1	10	1	40	1	3	1	3	1

a = 3052   b = 474   c = 854   d = 131   e = 170   f = 1051
g = 182   h = 326   i = 135   j = 13977   k = 451   m = 49968
n = 739   p = 121

Table II

$\sqrt[3]{5}$

FREQUENCY COUNTS

1	433	17	3	41	1	75	1
2	180	18	1	42	2	81	1
3	95	19	1	45	2	121	1
4	50	20	3	47	1	131	1
5	46	21	4	48	2	135	1
6	25	23	4	49	1	170	1
7	19	25	3	50	1	182	1
8	19	28	1	51	2	326	1
9	16	29	2	52	1	451	1
10	12	31	1	53	2	474	1
11	4	33	2	55	1	739	1
12	8	34	1	59	1	854	1
13	13	35	2	60	1	1051	1
14	2	37	3	61	2	3052	1
15	4	38	1	63	1	13977	1
16	2	40	2	72	1	49968	1

Table III

$\sqrt[3]{5}$

n	a_n	q_n	r_n
19	3052	4.28 E 11	162.730
74	474	1.23 E 40	5.511
92	60	9.82 E 49	0.548
104	61	2.43 E 56	0.491
215	81	1.88 E 109	0.331
226	52	2.20 E 116	0.200
277	51	3.15 E 140	0.162
279	854	5.44 E 143	2.636
372	131	1.57 E 183	0.315
375	170	1.07 E 186	0.402
378	51	1.34 E 189	0.120
388	59	4.04 E 195	0.134
406	1051	1.00 E 205	2.260
415	61	2.21 E 210	0.127
457	182	2.35 E 230	0.348
464	50	6.48 E 233	0.095
487	326	5.54 E 246	0.581
495	135	8.42 E 251	0.236
520	55	9.33 E 265	0.091
577	72	2.06 E 291	0.108
691	13977	2.20 E 345	17.792
762	75	1.46 E 379	0.087
765	63	7.47 E 381	0.073
809	451	4.93 E 401	0.491
813	49968	1.73 E 407	53.913
847	53	3.15 E 423	0.055
871	739	8.92 E 438	0.737
927	121	7.42 E 462	0.114
970	53	9.37 E 484	0.048

N 2000.5 = 3133 5483

Table I

$\sqrt[3]{7}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	1	10	2	16	2	1	4	2	1	21	1	3	5	1	2	1	1	2	11
20	5	1	3	1	2	27	4	1	*a	8	1	2	1	1	3	1	3	2	6	4
40	1	2	1	5	1	1	2	1	1	1	3	2	8	1	2	2	4	5	1	1
60	36	1	1	1	1	2	1	2	31	2	1	1	7	1	1	1	1	6	7	6
80	5	7	1	6	1	6	9	6	*b	5	33	8	2	1	1	1	6	2	1	1
100	1	1	2	13	7	1	1	1	17	3	2	1	1	1	23	*c	1	*d	1	72
120	1	4	6	6	1	11	6	12	7	3	5	1	4	6	2	2	1	2	4	4
140	1	1	9	11	6	2	1	15	6	2	1	1	2	1	3	5	1	2	1	1
160	1	1	1	15	1	35	1	2	1	2	4	2	1	11	2	1	1	1	1	15
180	4	1	1	2	22	2	1	31	2	2	1	1	3	1	10	2	1	2	1	1
200	2	1	1	8	1	5	1	1	6	2	1	14	1	4	7	5	2	6	6	1
220	4	1	1	3	2	10	1	3	2	16	2	34	1	1	1	18	3	1	7	1
240	3	12	1	1	15	3	2	1	2	2	1	2	2	17	3	2	2	3	5	8
260	1	56	1	2	1	25	1	3	1	3	3	8	20	8	1	3	1	1	1	8
280	1	4	12	1	3	6	3	1	3	7	19	3	1	1	13	1	1	1	1	10
300	2	2	1	1	3	15	1	4	1	*e	1	7	1	8	2	1	1	2	13	3
320	7	56	3	2	4	2	4	2	5	1	1	9	3	1	1	7	4	1	6	1
340	47	1	2	31	6	2	4	1	4	7	9	1	1	3	9	2	1	14	1	3
360	1	5	5	1	6	15	1	5	1	*f	6	1	1	11	4	1	6	8	19	1
380	1	1	8	2	6	4	19	3	14	14	3	1	11	1	6	1	1	2	2	1
400	1	1	1	5	2	1	1	1	1	4	2	3	2	1	1	1	1	1	2	3
420	1	2	1	4	1	2	12	1	1	13	3	1	1	2	*g	4	1	1	1	14
440	1	10	1	*h	7	60	9	32	1	6	2	13	1	1	1	2	1	98	1	1
460	1	2	1	2	1	*i	1	4	3	2	2	4	4	1	2	35	2	25	8	3
480	3	1	1	6	1	1	1	10	1	2	3	2	4	2	2	5	2	6	2	2
500	1	1	1	10	1	1	1	6	1	35	30	1	2	1	1	1	3	2	1	3
520	4	1	9	1	1	9	6	1	1	15	2	1	5	1	3	3	1	1	2	5
540	2	60	3	23	1	2	1	1	1	4	11	9	13	1	4	1	*j	2	1	11
560	3	2	1	1	5	2	52	1	4	1	8	2	3	2	3	25	1	14	1	1
580	1	7	8	1	2	11	15	1	2	14	2	1	6	1	2	7	1	1	2	2
600	1	1	1	2	1	1	1	1	1	9	*k	3	3	2	5	1	1	1	1	3
620	1	8	1	1	1	11	2	3	2	1	1	3	4	1	3	1	4	1	1	6
640	19	1	1	2	1	1	1	1	1	3	8	7	99	1	41	1	11	51	1	1
660	7	17	2	1	7	1	5	1	4	15	1	1	1	3	1	6	2	1	69	1
680	3	1	18	2	2	2	4	2	13	11	1	6	1	2	1	14	3	3	1	3
700	7	1	12	2	78	2	1	1	1	1	*m	5	1	10	5	2	1	2	5	1
720	6	3	2	2	10	13	4	27	1	2	1	1	44	19	4	3	1	3	1	1
740	2	2	5	3	1	1	1	1	1	1	1	2	1	1	1	11	1	8	6	1
760	7	1	1	2	19	1	5	3	1	1	3	1	3	1	1	44	1	34	1	5
780	3	8	1	2	3	*n	11	1	4	10	3	1	1	2	2	2	18	16	6	3
800	63	1	1	5	1	19	24	1	1	7	53	1	1	4	2	1	1	*p	2	1
820	5	2	7	5	4	1	1	3	1	30	1	1	1	2	1	7	1	7	1	1
840	24	2	1	4	1	1	3	2	1	3	21	1	1	1	73	1	4	2	1	1
860	4	9	1	2	4	2	3	1	1	3	7	2	11	2	1	4	8	8	15	11
880	18	17	1	1	2	7	43	2	3	1	1	6	2	1	10	1	5	3	3	3
900	8	22	4	3	1	4	4	4	6	1	7	1	1	9	1	18	3	4	1	1
920	2	1	2	7	1	2	2	1	15	1	1	2	5	1	2	52	3	1	87	1
940	3	1	1	1	6	10	1	1	1	3	2	4	1	6	1	2	1	3	2	3
960	1	1	1	5	2	1	1	8	1	1	8	1	2	4	1	*q	2	8	7	2
980	84	4	1	11	2	2	12	3	1	1	1	3	4	12	2	1	9	1	72	2

a = 282 b = 104 c = 277 d = 429 e = 303 f = 341
g = 110 h = 197 i = 118 j = 133 k = 689 m = 115
n = 111 p = 202 q = 628

Table II

$\sqrt[3]{7}$

FREQUENCY COUNTS

1	409	18	5	43	1	104	1
2	161	19	7	44	2	110	1
3	88	20	1	47	1	111	1
4	55	21	2	51	1	115	1
5	34	22	2	52	2	118	1
6	40	23	2	53	1	133	1
7	29	24	2	56	2	197	1
8	24	25	3	60	2	202	1
9	13	27	2	63	1	277	1
10	12	30	2	69	1	282	1
11	17	31	3	72	2	303	1
12	7	32	1	73	1	341	1
13	8	33	1	78	1	429	1
14	8	34	2	84	1	628	1
15	11	35	3	87	1	689	1
16	3	36	1	98	1		
17	4	41	1	99	1		

Table III

$\sqrt[3]{7}$

n	a_n	q_n	r_n
29	282	6.14 E 15	9.209
89	104	2.74 E 44	1.066
116	277	7.19 E 59	2.096
118	429	3.10 E 62	3.120
120	72	2.27 E 64	0.507
262	56	1.28 E 135	0.185
310	303	4.11 E 160	0.834
322	56	7.57 E 167	0.147
370	341	2.07 E 195	0.770
435	110	2.82 E 226	0.214
444	197	1.46 E 232	0.374
446	60	6.16 E 234	0.112
458	98	4.21 E 242	0.178
466	118	2.06 E 246	0.211
542	60	2.86 E 283	0.093
557	133	1.21 E 293	0.200
567	52	4.71 E 298	0.077
611	689	1.20 E 321	0.940
653	99	6.74 E 339	0.127
658	51	1.79 E 344	0.065
679	69	1.36 E 355	0.086
705	78	9.95 E 369	0.093
711	115	1.50 E 373	0.135
786	111	4.29 E 411	0.118
801	63	9.92 E 421	0.065
811	53	5.06 E 428	0.054
818	202	5.07 E 432	0.204
855	73	2.51 E 450	0.071
936	52	3.44 E 494	0.046
939	87	1.21 E 497	0.077
976	628	2.69 E 513	0.535
981	84	5.88 E 517	0.071
999	72	1.14 E 528	0.060

Table I

$$\alpha_1 = 2\cos\frac{2\pi}{7}\,,\ \text{Root of } x^3 + x^2 - 2x - 1$$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	4	20	2	3	1	6	10	5	2	2	1	2	2	1	18	1	1	3	2
20	1	2	1	2	1	39	2	1	1	1	13	1	2	1	30	1	1	1	3	2
40	5	4	1	5	1	5	1	2	1	1	94	6	2	19	11	1	60	1	1	50
60	2	1	1	8	53	1	3	1	6	3	2	1	5	1	1	3	4	*a	1	2
80	1	3	3	7	9	1	2	10	3	1	22	1	*b	3	32	1	2	1	2	4
100	2	1	2	2	62	2	1	1	8	1	14	5	6	5	1	1	8	1	7	*c
120	1	1	2	2	1	2	2	2	2	30	3	1	13	1	19	3	1	4	1	1
140	2	1	33	1	10	1	13	2	26	1	1	1	9	1	9	1	6	13	1	5
160	1	1	1	6	1	9	1	7	1	3	1	1	1	12	1	1	4	3	1	8
180	1	2	2	2	1	2	4	1	1	1	1	3	1	19	1	3	3	2	2	1
200	13	3	4	1	1	1	3	1	1	1	2	1	3	19	1	3	9	3	2	4
220	1	3	1	6	1	25	20	1	1	2	1	2	5	1	1	1	14	1	13	1
240	1	1	55	8	1	1	24	17	1	11	17	1	4	1	1	9	1	1	1	3
260	1	1	1	1	20	4	1	45	1	2	4	1	1	2	1	1	2	11	*d	1
280	4	1	2	1	2	*e	1	3	5	3	7	3	25	2	3	1	1	2	1	1
300	1	1	5	1	1	5	2	1	1	1	1	1	1	3	2	1	4	2	1	5
320	1	1	6	1	30	1	62	1	36	11	7	1	21	1	19	1	15	1	2	12
340	5	5	9	2	1	5	1	1	1	2	6	1	3	1	1	3	*f	8	30	2
360	1	1	1	3	1	3	1	1	44	2	19	1	3	1	1	17	20	1	4	3
380	97	2	1	10	4	2	1	5	1	1	4	1	3	18	1	1	1	4	14	2
400	4	13	3	2	9	2	1	2	1	2	1	4	1	22	1	1	3	1	16	61
420	1	2	3	2	5	1	1	2	1	1	7	3	11	2	1	4	2	3	1	2
440	2	1	27	4	1	3	5	17	2	2	10	2	1	2	15	1	37	5	7	1
460	24	4	56	2	2	4	1	1	1	4	3	1	2	8	3	3	2	4	1	1
480	2	14	*g	1	16	1	5	1	1	1	1	*h	1	3	1	*i	1	1	13	2
500	2	3	2	1	1	1	2	1	11	1	8	4	2	72	3	1	5	7	1	3
520	7	2	1	4	4	2	9	2	1	2	3	10	1	1	2	1	13	1	5	1
540	4	4	2	1	1	6	3	1	3	1	1	5	2	1	49	1	10	2	1	11
560	1	3	4	1	6	1	2	1	2	1	2	2	1	13	1	3	2	2	3	2
580	1	2	*j	1	3	2	2	1	3	3	9	1	3	*k	3	2	3	12	6	11
600	3	2	1	1	1	2	2	2	1	4	1	3	4	7	1	1	3	1	1	1
620	7	1	16	19	9	1	6	8	1	75	1	2	3	7	3	1	1	2	4	2
640	2	4	2	1	1	27	1	1	1	9	15	1	9	3	12	1	13	3	2	16
660	1	1	18	2	3	1	6	1	7	4	1	1	1	1	2	2	4	4	1	3
680	8	1	19	3	21	1	3	1	4	3	3	1	2	2	1	*m	4	7	1	*n
700	1	1	2	1	6	1	2	1	1	3	4	1	2	2	2	22	3	1	9	4
720	3	1	5	1	1	3	5	20	1	12	2	1	1	1	1	2	87	2	2	2
740	59	1	1	2	1	4	17	3	1	1	1	1	2	3	2	1	1	2	3	3
760	1	*p	4	11	19	2	2	1	5	2	3	1	4	1	1	1	1	4	3	1
780	6	1	12	2	7	1	5	4	9	3	3	2	2	1	1	2	4	2	4	3
800	5	3	1	1	5	1	1	4	1	1	20	1	1	1	6	1	6	2	3	1
820	3	1	2	9	1	1	1	10	10	*q	1	1	10	1	3	1	14	1	2	2
840	5	1	1	2	3	2	6	1	3	1	12	1	25	9	*r	1	1	3	1	26
860	34	2	2	5	3	1	2	5	2	3	2	1	1	2	1	8	2	1	1	5
880	3	1	2	9	32	1	1	3	4	1	3	1	1	3	1	1	20	2	2	1
900	3	10	57	1	*s	20	1	1	1	66	1	26	1	4	4	6	5	50	1	1
920	5	3	1	1	6	21	4	4	1	1	1	1	2	7	5	3	9	5	3	1
940	4	2	1	2	1	17	1	59	3	1	8	1	1	10	3	4	2	5	1	1
960	1	2	1	14	5	7	1	1	6	46	1	2	4	6	3	1	3	8	24	7
980	1	1	2	1	3	11	4	1	14	1	13	2	1	2	2	1	7	2	2	1

a = 636 b = 119 c = 425 d = 202 e = 136 f = 699
g = 424 h = 165 i = 114 j = 283 k = 267 m = 716
n = 108 p = 704 q = 904 r = 124 s = 152

Table II

$$\alpha_1 = 2\cos\frac{2\pi}{7}, \text{ Root of } x^3 + x^2 - 2x - 1$$

FREQUENCY COUNTS

1	401	15	3	33	1	59	2	136	1
2	168	16	4	34	1	60	1	152	1
3	109	17	6	36	1	61	1	165	1
4	60	18	3	37	1	62	2	202	1
5	40	19	9	39	1	66	1	267	1
6	23	20	8	44	1	72	1	283	1
7	19	21	3	45	1	75	1	424	1
8	13	22	3	46	1	87	1	425	1
9	19	24	3	49	1	94	1	636	1
10	12	25	3	50	2	97	1	699	1
11	10	26	3	53	1	108	1	704	1
12	7	27	2	55	1	114	1	716	1
13	12	30	4	56	1	119	1	904	1
14	7	32	2	57	1	124	1		

Table III

$$\alpha_1 = 2\cos\frac{2\pi}{7}, \text{ Root of } x^3 + x^2 - 2x - 1$$

n	a_n	q_n	r_n
51	94	1.34 E 24	1.854
57	60	2.49 E 29	0.958
60	50	2.53 E 31	0.739
65	53	5.83 E 34	0.698
78	636	2.51 E 42	6.978
93	119	3.20 E 51	1.054
105	62	1.96 E 58	0.480
120	425	1.61 E 68	2.815
243	55	7.83 E 124	0.196
279	202	4.61 E 144	0.617
286	136	3.97 E 148	0.404
327	62	3.30 E 166	0.166
357	699	1.99 E 185	1.665
381	97	1.44 E 200	0.213
420	61	2.64 E 220	0.121
463	56	4.85 E 244	0.101
483	424	6.15 E 255	0.728
492	165	6.02 E 260	0.278
496	114	3.47 E 263	0.191
514	72	4.03 E 272	0.116
583	283	3.59 E 305	0.406
594	267	1.45 E 312	0.375
630	75	4.20 E 331	0.100
696	716	2.26 E 366	0.856
700	108	8.11 E 369	0.129
737	87	2.72 E 387	0.098
741	59	1.95 E 390	0.066
762	704	2.68 E 400	0.770
830	904	5.77 E 434	0.909
855	124	2.41 E 448	0.121
903	57	3.80 E 472	0.053
905	152	5.92 E 474	0.141
910	66	2.45 E 478	0.061
918	50	2.34 E 484	0.045
948	59	1.04 E 500	0.052

N1526.5 = ~~3135~~ 39922

Table I

α_2, Root of $x^5 - x - 1$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	5	1	42	1	3	24	2	2	1	16	1	11	1	1	2	31	1	12	5
20	1	7	11	1	4	1	4	2	2	3	4	2	1	1	11	1	41	12	1	8
40	1	1	1	1	1	9	2	1	5	4	1	25	4	6	11	1	4	1	6	1
60	1	1	2	2	2	4	11	1	4	1	3	2	8	1	3	3	6	21	11	2
80	1	1	10	2	1	3	2	8	1	10	4	3	1	1	1	1	2	1	1	18
100	7	4	1	2	2	6	1	1	2	2	6	20	1	43	3	2	4	2	1	1
120	2	3	1	4	4	1	3	1	1	12	1	2	2	3	2	14	1	5	1	7
140	1	2	2	10	2	2	2	1	38	1	59	2	1	4	1	4	2	4	23	13
160	1	1	1	3	1	32	2	1	3	6	4	1	4	1	1	1	1	91	1	7
180	2	8	1	18	2	2	1	1	28	2	1	12	1	3	4	55	1	1	2	2
200	6	1	3	1	16	1	2	1	1	*a	1	1	4	3	2	1	2	1	4	1
220	1	2	43	3	3	1	55	1	1	2	10	1	2	1	12	12	36	1	8	1
240	18	1	3	1	1	1	6	1	24	2	1	1	1	6	1	3	1	1	11	1
260	4	2	1	1	1	2	2	1	73	1	1	4	*b	54	1	8	4	2	3	4
280	1	1	1	1	99	4	2	2	1	4	1	1	1	10	1	1	1	1	1	1
300	1	2	1	5	3	21	5	1	6	2	1	3	2	1	1	4	1	1	1	85
320	8	1	1	30	1	3	2	3	1	1	3	1	1	1	85	6	1	1	3	2
340	1	10	1	1	*c	3	1	1	17	1	12	1	1	3	5	2	1	3	1	3
360	1	5	1	1	1	*d	1	2	1	2	1	9	3	1	1	3	1	4	3	7
380	1	69	5	1	2	2	3	1	1	8	17	13	1	2	3	2	11	1	8	2
400	4	5	1	2	1	5	1	1	1	5	2	1	10	1	1	1	1	1	6	3
420	3	1	5	2	2	6	4	1	10	1	2	1	1	1	4	2	1	1	2	1
440	4	1	2	4	5	1	1	3	1	5	6	1	4	1	*e	3	18	2	11	9
460	9	2	20	1	10	2	4	1	1	1	5	3	2	2	2	4	3	1	1	8
480	1	7	4	1	3	12	16	1	1	2	2	2	2	3	5	2	1	3	1	16
500	2	1	1	2	4	1	3	5	2	12	1	1	1	12	1	2	26	21	7	2
520	1	2	8	2	2	1	1	1	2	2	1	1	1	3	1	1	1	1	39	4
540	1	29	18	1	8	13	3	1	1	1	1	1	8	1	4	1	3	2	2	2
560	1	5	2	5	1	5	2	8	8	2	8	5	1	4	3	2	2	2	3	3
580	7	2	4	4	2	18	6	*f	6	32	5	13	2	3	6	1	5	2	1	1
600	1	1	1	3	12	2	1	1	2	1	1	48	1	1	1	13	1	5	1	4
620	1	1	5	1	1	3	1	2	1	21	2	2	3	12	1	3	1	1	3	2
640	3	1	*g	11	5	1	1	12	2	2	2	2	3	3	14	1	42	17	1	1
660	1	2	1	2	1	1	5	2	8	1	2	18	2	27	1	14	1	1	3	1
680	1	4	2	3	3	3	1	2	1	9	1	1	1	1	4	1	17	4	3	12
700	1	25	15	5	1	2	2	6	1	7	7	5	1	5	1	7	1	2	1	1
720	1	*h	1	1	1	2	2	1	1	34	4	5	4	16	3	4	1	1	1	10
740	46	2	1	1	1	5	1	1	2	1	7	10	1	3	2	1	1	2	1	7
760	4	*i	1	4	6	1	4	1	1	8	1	1	15	2	3	16	7	1	6	1
780	3	1	1	1	7	1	1	1	1	3	4	1	1	10	1	1	4	2	1	47
800	1	3	3	6	1	*j	74	14	2	24	22	*k	3	9	2	5	3	2	4	2
820	1	19	2	1	2	8	4	2	5	7	4	1	1	4	3	25	1	1	*m	*n
840	2	44	3	1	50	1	1	2	1	10	1	15	1	1	*p	5	2	1	3	37
860	5	7	1	4	4	1	26	2	2	*q	1	7	2	1	11	2	1	2	1	1
880	1	5	1	6	1	1	4	14	3	1	2	1	1	12	1	9	52	1	9	6
900	2	2	4	1	33	3	3	*r	1	1	23	7	1	2	9	1	7	1	2	1
920	1	1	7	4	1	1	1	17	9	2	3	1	14	35	1	1	1	6	9	12
940	1	4	2	*s	1	5	3	1	1	1	2	5	3	7	1	32	8	1	6	1
960	1	3	*t	1	25	1	1	1	26	3	1	3	1	1	7	2	17	6	1	1
980	6	4	1	6	2	23	1	3	2	5	8	1	3	10	1	30	1	13	1	2

a = 761	b = 195	c = 166	d = 264	e = 701	f = 11644
g = 169	h = 673	i = 457	j = 409	k = 274	m = 174
n = 124	p = 172	q = 1033	r = 110	s = 684	t = 1292

Table II

α_2, Root of $x^5 - x - 1$

FREQUENCY COUNTS

1	406	16	6	31	1	48	1	169	1
2	162	17	6	32	3	50	1	172	1
3	89	18	7	33	1	52	1	174	1
4	67	19	1	34	1	54	1	195	1
5	41	20	2	35	1	55	2	264	1
6	28	21	4	36	1	59	1	274	1
7	23	22	1	37	1	69	1	409	1
8	21	23	3	38	1	73	1	457	1
9	11	24	3	39	1	74	1	673	1
10	14	25	4	41	1	85	2	684	1
11	11	26	3	42	2	91	1	701	1
12	16	27	1	43	2	99	1	761	1
13	6	28	1	44	1	110	1	1033	1
14	6	29	1	46	1	124	1	1292	1
15	3	30	2	47	1	166	1	11644	1

Table III

α_2, Root of $x^5 - x - 1$

n	a_n	q_n	r_n
151	59	2.97 E 77	0.344
178	91	3.13 E 91	0.444
196	55	1.29 E 102	0.239
210	761	5.27 E 109	3.096
227	55	1.58 E 118	0.208
269	73	2.00 E 138	0.235
273	195	3.54 E 141	0.609
274	54	1.91 E 143	0.166
285	99	1.35 E 149	0.294
320	85	8.07 E 163	0.230
335	85	3.32 E 171	0.219
345	166	1.87 E 177	0.413
366	264	4.47 E 187	0.621
382	69	1.92 E 195	0.157
455	701	1.44 E 229	1.347
588	11644	6.58 E 300	17.042
643	169	7.49 E 327	0.226
722	673	8.41 E 369	0.797
762	457	1.47 E 391	0.511
806	409	1.39 E 413	0.434
807	74	1.03 E 415	0.078
812	274	4.43 E 421	0.284
839	174	8.98 E 437	0.174
840	124	1.11 E 440	0.123
845	50	2.03 E 444	0.050
855	172	9.54 E 449	0.167
870	1033	3.09 E 461	0.979
897	52	3.27 E 474	0.048
908	110	2.20 E 482	0.100
944	684	3.68 E 503	0.594
963	1292	1.91 E 515	1.096

39923

Table I

$$\alpha_3 = \sqrt[3]{2} + \sqrt{3}\,,\quad x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	2	1	*a	1	1	3	1	1	5	7	2	3	2	4	1	18	5	1	13	3
20	3	3	4	1	69	2	1	1	7	1	1	3	1	1	13	2	5	2	1	3
40	1	2	38	3	1	2	1	1	2	1	5	1	1	*b	1	1	6	1	2	5
60	1	1	9	4	1	5	2	1	4	5	1	1	18	3	3	2	24	3	1	1
80	1	2	74	3	2	4	3	1	1	10	1	1	1	1	4	1	1	1	3	7
100	8	*c	4	1	4	1	1	2	1	5	1	2	3	1	23	18	4	1	2	1
120	85	1	2	1	2	1	8	1	1	1	22	3	1	3	1	1	8	3	15	30
140	1	7	1	1	1	11	4	1	19	1	1	1	3	6	1	44	3	8	3	1
160	1	1	10	4	1	8	3	5	16	6	3	1	2	12	1	2	3	2	1	3
180	9	1	5	2	4	1	3	2	26	1	2	1	1	2	2	4	2	1	3	2
200	4	5	1	4	1	2	1	4	*d	1	1	8	4	1	1	9	1	1	2	5
220	1	4	2	2	1	29	4	1	*e	3	61	1	4	15	1	3	23	[illegible]	5	1
240	1	1	2	3	2	3	6	1	8	1	2	2	1	1	10	1	1	3	3	*f
260	1	1	1	1	1	3	7	8	1	42	1	3	1	1	2	1	2	6	2	1
280	1	1	1	7	16	1	1	1	1	3	37	1	7	38	63	3	1	14	6	1
300	1	4	1	1	2	4	1	6	1	1	1	1	3	1	30	4	1	4	1	8
320	10	4	3	*g	25	2	1	2	1	1	*h	1	1	1	2	2	6	2	9	1
340	13	1	2	4	4	1	1	19	1	1	3	2	3	1	2	1	1	1	4	1
360	7	2	1	*i	8	4	20	1	2	1	3	1	1	94	2	1	3	4	1	3
380	1	7	1	3	1	9	9	1	4	2	4	2	35	1	2	2	1	2	1	1
400	3	12	4	1	1	2	6	1	1	1	1	2	3	13	1	1	5	1	3	7
420	7	2	4	4	3	1	1	1	1	3	12	10	2	2	1	1	1	1	1	3
440	6	1	2	6	28	2	1	1	1	2	1	1	1	2	1	1	10	1	5	3
460	2	3	1	1	3	2	2	8	1	13	4	1	1	7	1	1	2	2	10	4
480	1	1	5	1	8	2	1	4	4	1	1	1	8	12	2	5	3	18	4	27
500	2	3	1	1	1	1	4	3	2	1	7	1	1	1	8	5	1	5	1	5
520	2	1	4	1	14	14	1	1	1	1	2	1	1	22	7	1	1	5	17	1
540	2	4	1	15	1	3	1	1	1	11	1	2	16	1	1	3	1	1	1	9
560	1	3	30	4	3	4	36	1	6	2	2	1	36	2	2	1	5	7	1	1
580	99	2	16	1	37	1	2	1	1	3	1	4	1	1	5	2	2	1	3	3
600	1	3	2	4	1	2	2	31	1	11	1	1	24	1	2	8	3	1	*j	1
620	*k	3	19	2	2	8	1	1	11	2	1	7	1	1	1	7	1	1	5	3
640	3	1	25	1	8	2	2	1	6	1	1	2	1	2	1	1	4	1	1	10
660	22	1	37	1	19	2	1	17	1	38	2	3	8	1	8	2	30	1	2	2
680	5	2	2	3	1	2	1	7	1	3	1	1	12	1	11	3	3	*m	6	1
700	30	1	1	2	1	1	7	1	5	7	1	75	1	12	1	2	1	1	7	1
720	1	2	2	1	1	7	1	1	3	1	1	27	10	4	1	6	2	1	*n	1
740	2	2	1	1	4	5	7	3	17	21	1	1	58	13	33	2	4	1	5	3
760	12	1	16	3	3	7	*p	1	1	13	2	*q	1	7	2	1	3	1	1	1
780	1	1	2	1	2	1	1	2	7	3	1	3	34	13	10	1	1	1	3	1
800	1	32	1	*r	3	55	3	2	1	6	1	3	1	2	2	1	2	5	1	7
820	1	34	1	5	1	13	1	2	8	1	9	5	1	21	3	2	4	1	1	1
840	1	1	1	5	7	1	1	2	1	2	3	5	28	1	1	11	1	4	1	3
860	2	47	2	3	14	1	1	2	29	1	1	1	7	1	3	1	3	1	2	1
880	8	1	1	1	2	1	4	2	2	2	1	1	2	1	2	13	2	1	50	13
900	23	1	2	5	6	1	2	1	2	1	53	1	6	3	3	3	23	1	1	1
920	5	1	11	1	4	5	*s	2	9	1	27	1	15	2	1	29	1	3	2	2
940	2	3	6	2	80	3	1	9	1	3	9	1	1	2	1	3	11	8	17	1
960	3	1	1	4	8	2	3	1	*t	6	1	1	2	1	*u	3	38	2	1	2
980	3	1	2	2	3	1	9	5	1	8	2	7	2	1	1	1	5	8	1	10

a = 123, b = 1446, c = 126, d = 121, e = 154, f = 452
g = 315, h = 135, i = 103, j = 120, k = 331, m = 184
n = 133, p = 430, q = 298, r = 150, s = 208, t = 186
u = 138

Table II

$$\alpha_3 = \sqrt[3]{2} + \sqrt{3}\,,\ x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$$

FREQUENCY COUNTS

1	418	20	1	42	1	123	1
2	156	21	2	44	1	126	1
3	105	22	3	47	1	133	1
4	56	23	4	50	1	135	1
5	38	24	2	53	1	138	1
6	20	25	2	55	1	150	1
7	30	26	1	58	1	154	1
8	25	27	3	61	1	184	1
9	12	28	2	63	1	186	1
10	11	29	3	69	1	208	1
11	8	30	5	74	1	298	1
12	7	31	1	75	1	315	1
13	11	32	1	80	1	331	1
14	4	33	1	85	1	430	1
15	4	34	2	94	1	452	1
16	5	35	1	99	1	1446	1
17	4	36	2	103	1		
18	4	37	3	120	1		
19	4	38	4	121	1		

Table III

$$\alpha_3 = \sqrt[3]{2} + \sqrt{3}\,,\ x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$$

n	a_n	q_n	r_n
3	123	1.24 E 2	************
25	69	1.45 E 14	2.462
54	1446	3.93 E 28	24.700
83	74	2.19 E 43	0.779
102	126	1.22 E 53	1.074
121	85	1.36 E 63	0.608
209	121	8.04 E 107	0.497
229	154	3.52 E 118	0.578
231	61	6.49 E 120	0.224
260	452	9.82 E 135	1.473
295	63	1.06 E 154	0.180
324	315	6.59 E 169	0.818
331	135	4.31 E 174	0.341
364	103	4.51 E 189	0.240
374	94	1.00 E 196	0.212
581	99	2.74 E 294	0.148
619	120	6.52 E 313	0.168
621	331	2.18 E 316	0.459
698	184	9.56 E 356	0.226
712	75	7.91 E 364	0.091
739	133	9.57 E 377	0.154
753	58	3.86 E 386	0.066
767	430	2.47 E 398	0.472
772	298	4.13 E 402	0.324
804	150	1.49 E 418	0.158
806	55	2.48 E 420	0.057
899	50	2.45 E 463	0.048
911	53	6.77 E 470	0.050
927	208	5.44 E 480	0.189
945	80	9.75 E 491	0.071
969	186	4.09 E 505	0.161
975	138	2.61 E 509	0.119

The program to do the calculation was written in Fortran, using machine-language subroutines for multiple-precision integer arithmetic to handle the coefficients of the polynomials. The calculation of a_n was done in floating-point arithmetic (approximately 14 significant digits), using a floating-point approximation to $P_n(x)$ (suitably scaled). This procedure avoids the use of multiple presicion arithmetic in any trial-and-error steps, and so makes for greater efficiency. One could be even more efficient, using an idea suggested by Lehmer [5], and compute several successive partial quotients from an approximation to $P_n(x)$. It is possible to find $P_{n+m}(x)$ from $P_n(x)$ and $a_n, a_{n+1}, \ldots, a_{n+m-1}$ with less multiple-precision calculation than is needed to find all the intervening polynomials explicitly. The additional complication in the program, however, did not seem worth while, since the results given here were obtained by the simpler method in about 6 minutes on an IBM 360/91. A listing of the actual program may be obtained on request from Trotter.

References

[1] *Churchhouse* and *Muir*, Continued fractions, algebraic numbers, and modular invariants, J. Inst. Math. App. 5 (1969), 318—328.

[2] *A. Y. Khinchin*, Continued fractions, Chicago 1964.

[3] *S. Lang*, Report on diophantine approximations, Bull. Math. Soc. France **93** (1965), 177—192.

[4] *S. Lang*, Introduction to diophantine approximations, Reading 1967.

[5] *D. Lehmer*, Euclid's algorithm for large numbers, Am. Math. Monthly **45** (1938), 227—233.

[6] *H. Stark*, An explanation of some exotic continued fractions found by Brillhart, to appear.

Fine Hall, Princeton University, Princeton, N. J. 08540 USA

Eingegangen 3. Juni 1971