$$s_{jk} = \sum_{i=j}^{k} x_i$$

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Sum Triangles of Natural Numbers Having Minimum Top

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ABSTRACT

Golomb's results [3], on sum triangles *(difference sets)* are herein improved and extended. Equivalent problems have also been considered by B. Lindström [8]. Exact values of minimum tops for sum triangles of size n, $1 \le n < 14$ are found and an *extreme* sum triangle *(one having a minimum top)* is given for each case considered here.

Introduction.

Let $X = (x_1, x_2, \ldots, x_n)$ be a sequence of natural numbers.

Define, for $1 \le j \le k \le n$, $s_{jk} = \sum_{i=j}^{k} x_i$

Clearly, $s_{jj} = x_j$, $1 \le j \le n$.

It is convenient to display the s_{jk}, $1 \le j \le k \le n$, in the form c triangle T as shown in Figure 1.

$s_{1,n}$

$s_{1,n-1}$ $s_{2,n}$

s_{13} . . . $s_{n-2,n}$

s_{12} s_{23} $\cdots$ $s_{n-2,n-1}$ $s_{n-1,n}$

s_{11} s_{22} s_{33} $\cdots$ $s_{n-1,n-1}$ $s_{n,n}$

Figure 1.

T is called a *sum triangle* if and only if all $\binom{n+1}{2}$ numbers $1 \le j \le k \le n$, are distinct or $|T| = \binom{n+1}{2}$. We note that $s_{jk} = \sum_{i=j}^{k}$ is the sum of the corresponding entries of the first row of the subtria whose top is s_{jk}.

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$\begin{matrix} & 3 & \\ 1 & & 2 \end{matrix}$ and $\begin{matrix} & & 6 & & \\ & 4 & & 5 & \\ 1 & & 3 & & 2 \end{matrix}$ are examples of sum triangles.

Let R_i be the sequence $(s_{1i}, s_{2,i+1}, \ldots, s_{n-i+1,n})$. Then, R_i, $1 \le i \le n$, is called the ith row of T and $s_{1,n} \in R_n$ is called the *top of* T.

The open problem which has resisted the efforts of many mathematicians is to determine the minimum top in a sum triangle for a given n. We denote this value by $\tau(n)$.

The triangle T can also be defined as a difference set of a given increasing sequence A of integers, say, $a_0 < a_1 < \cdots < a_n$, where for every $1 \le i \le n$, $a_i - a_{i-1} = x_i$. As before, the other differences form the other rows of the triangle T. In this case, T is called a difference triangle or a component of the system of difference sets, [1,2,6]. T is uniquely determined by A. However, the converse is not true; because addition of the same constant to each term of the sequence, or multiplying each term by -1 and then reversing the order, yields a sequence which has the same difference set. We consider the mirror image of T as being equivalent to T.

The first three values of τ follow readily from the definition. The values of τ for $n \in \{4,5,6\}$ were given by Golomb [3]. He did not prove them to be minimum. In [3], Golomb gave a construction which proved that $\tau(7) \le 36$, $\tau(8) \le 48$ and $\tau(9) \le 64$. The results herein improve those of Golomb.

Table 1 contains solutions for $n < 14$. It suffices to list only the first row of each sum triangle T, as T is completely determined by it.

n	$\tau(n)$	First Row of Sum Triangle
1	1	1
2	3	1 2
3	6	1 3 2
4	11	1 3 5 2
5	17	1 3 6 2 5
6	25	1 3 6 8 5 2
7	34	1 3 5 6 7 10 12
8	44	3 6 8 2 13 7 4 1
9	55	2 12 7 8 3 13 4 5 1
10	72	2 6 10 7 14 5 15 9 3 1
11	85	9 1 7 13 12 3 11 5 10 4 2
12	106	7 1 9 4 15 11 16 6 12 20 3 2
13	127	5 23 10 3 8 1 18 7 17 15 14 2 4

Table 1

For $n > 11$, the best known published estimates of $\tau(n)$ are due to Lindström [8].

His result, when translated into our notation yields:

$$n \le (\tau(n))^{1/2} + (\tau(n))^{1/4} + 1. \qquad (1)$$

The above inequality does not give satisfactory estimates for the tops of the sum triangles for the cases considered here, $3 < n < 14$. For $3 < n < 14$, the estimates given by (1) are denoted by t_1 and are listed in Table 2.

Notation:

$$\sigma(n) = \sum_{i=1}^{n} i; \quad S_i = \sum_{x \in R_i} x.$$

The following result is given in [2, Prop. 1.1].

Lemma 1: *In any sum triangle T,*

$$\sum_{i=1}^{k} S_i = \sum_{i=1}^{k} S_{n-i+1}, \quad 1 \le k \le n.$$

In order to improve the estimates of $\tau(n)$ given by (1), consider the following inequalities for a sum triangle of size n with top $t_2(n)$:

$$\sum_{i=1}^{2n-1} i = \frac{1}{2}(2n)(2n-1) \le \sum_{i=1}^{2} S_i \le 3t_2(n) - 3. \qquad (2)$$

$$\sum_{i=1}^{3n-3} i = \frac{1}{2}(3n-3)(3n-2) \le \sum_{i=1}^{3} S_i \le 6t_2(n) - 16. \qquad (3$$

$$\sum_{i=1}^{4n-6} i = \frac{1}{2}(4n-6)(4n-5) \le \sum_{i=1}^{4} S_i \le 10t_2(n) - 50. \qquad (4$$

(3) implies that $t_2(n) \ge \frac{17}{6} + \frac{1}{4}(n-1)(3n-2)$ and (4) implies tha $t_2(n) \ge 5{,}6 + \frac{1}{10}(2n-3)(4n-5)$. Since $t_2(n)$ must be an integer, one ca easily show that the best approximations to $\tau(n)$ using the above methoc for $4 \le n < 14$ are obtained from (3). These are listed in Table 2 as t_2.

Table 2 also includes the estimates t_3 of the tops for a given size n They are an improvement over those obtained for t_2. The methods use in obtaining t_3 are concretely described in the proofs which follow. Thes methods are a variation of those used for t_2. The values of $\tau(n$ $6 \le n < 14$ were calculated using t_3 and a simple computer program.

Remark:

For $i = 1,2,3$, $t_i(n)$ has the following property:
for a given n, $t_i(n) \le \tau(n)$, $i = 1,2,3$.

n	$t_1(n)$	$t_2(n)$	$t_3(n)$	$\tau(n)$
3	1	6	6	6
4	3	11	11	11
5	6	16	16	17
6	11	23	24	25
7	16	32	33	34
8	24	42	43	44
9	42	53	67	72
11	54	66	81	85
12	67	81	98	106
13	81	97	116	127

← 55

Table 2

Proof that $\tau(4) = 11$:

From Lemma 1, $S_1 + S_2 \ge \sigma(7) = 28$. Also,

$$S_1 + S_2 \le \tau(4) + [\tau(4)-1] + [\tau(4)-2] = 3\tau(4) - 3.$$

Thus, $3\tau(4) \ge 31$ which implies $\tau(4) > 10$.

Table 1 lists a sum triangle with $\tau(4) = 11$.

Proof that $\tau(5) = 17$:

From Lemma 1, $S_1 + S_2 = 3\tau(5) - (x_1+x_5)$. But $x_1 + x_5 \ge 3$ gives $S_1 + S_2 \le 3\tau(5) - 3$. Also, $S_1 + S_2 \ge \sigma(9) = 45$.

Thus, $45 \le 3\tau(5) - 3$, which implies $\tau(5) \ge 16$.

Suppose $\tau(5) = 16$. Without loss of generality one may assume that $x_1 = 1$ and $x_5 = 2$. It now follows that $x_2 + x_3 + x_4 = 13$. This implies that $\{x_2,x_3,x_4\} = \{3,4,6\}$. $x_2 \ne 3$ because $x_1 + x_2 \in R_2$ would be equal to 4 and $4 \in R_1$. Similarly $x_4 \ne 4$ because then $x_4 + x_5 = 6 \in R_2$ and $6 \in R_1$. Only two cases remain to be considered:

Case 1: $x_1 = 1$, $x_2 = 4$ and $x_5 = 2$.

Case 2: $x_1 = 1$, $x_2 = 6$ and $x_5 = 2$.

Case 2 is not possible because $x_1 + x_2 = 7$ and $x_3 + x_4 = 7$. In Case 1, $x_4 \ne 3$ because $x_1 + x_2 = 5$ and this must be distinct from $x_4 + x_5$. Thus, $x_3 = 3$ and $x_4 = 6$. Constructing the sum triangle leads to a contradiction.

Thus, $\tau(5) \ge 17$. Table 1 lists a sum triangle with $\tau(5) = 17$.

Proof that $\tau(6) = 25$:

Consider the sum $U = S_1 + S_2 + s_{13} + s_{46}$.

Clearly, $U \ge \sigma(13) = 91$. Using the properties of a sum triangle, one obtains $4\tau(6) - (x_1+x_6) \ge 91$, which implies

$$4\tau(6) \ge 91 + (x_1+x_6) \ge 94.$$

Since $\tau(6)$ is an integer, $\tau(6) \ge 24$.

In a sum triangle of size six *(having six rows)* $s_{13} - s_{46} = s_{16}$. It is known by use of the computer that there are only two incomplete perfect systems of difference sets having two components of size three and for which $s_{13} + s_{46} = 24$.

They are given by:

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    16          8              13         11
  13  12      7   6    and   12  10      7   6
 4   9   3   2  5  1        3   9   1   5  2  4
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None of the above systems and their mirror images can be completed to give a sum triangle for $\tau(6)$. Thus, $\tau(6) > 24$. Table 1 lists a sum triangles with $\tau(6) = 25$.

Proof that $\tau(7) = 34$:

Consider the following two sums:

$$S_1 + S_2 + s_{13} + s_{46}$$
$$= 4\tau(7) - (x_1+x_4+x_7) \ge \sigma(15) = 120 \qquad (5)$$

and

$$S_1 + S_2 + S_3 - s_{25} + s_{23} + s_{57}$$
$$= 6\tau(7) - 2(x_1+x_7) + x_4 \ge \sigma(19) = 190. \qquad (6)$$

Adding (5) and (6), one obtains

$$10\tau(7) - 3(x_1+x_7) \ge 310.$$

Thus, $10\tau(7) \ge 310 + 3(x_1+x_7)$. Since $x_1 + x_7 \ge 3$, then $10\tau(7) \ge 319$ or $\tau(7) \ge 32$. Suppose $\tau(7) = 32$. Then $320 \ge 310 + 3(x_1+x_7)$. This implies $10 \ge 3(x_1+x_7)$ or $x_1 + x_7 = 3$. Without loss of generality one may assume that $x_1 = 1$ and $x_7 = 2$. Using (5), one obtains

$$4\tau(7) - x_4 \ge 123.$$

Thus, $128 \ge x_4 + 123$ or $x_4 \le 5$. Using (6), one obtains

$$6\tau(7) + x_4 \geq 190 + 2(x_1+x_7).$$

Thus, $192 + x_4 \geq 196$ or $x_4 \geq 4$.

Consider the sum

$$S_1 + S_2 + S_3 + s_{14} + s_{47}$$
$$= 7\tau(7) - 3(x_1+x_7) - (x_2+x_6) + x_4 \geq \sigma(20)$$
$$= 210. \qquad (7)$$

Adding (5) and (7) yields

$$11\tau(7) - 4(x_1+x_7) - (x_2+x_6) \geq 330.$$

Thus, $11\tau(7) \geq 330 + 4(x_1+x_7) + (x_2+x_6)$. Since $\tau(7) = 32$ and $x_1 + x_7 = 3$ one has $10 \geq x_2 + x_6$. Thus, $x_2 + x_6 \in \{8,9,10\}$. Also, $x_4 \in \{4,5\}$. The above yields the following twelve cases:

Case	X_1	X_2	X_4	X_6	X_7
1	1	3	4	5	2
2	1	5	4	3	2
3	1	3	4	6	2
4	1	6	4	3	2
5	1	3	4	3	2
6	1	7	4	3	2
7	1	3	5	6	2
8	1	6	5	3	2
9	1	3	5	7	2
10	1	7	5	3	2
11	1	4	5	6	2
12	1	6	5	4	2

Cases 1, 3 and 5 are not possible because $x_1 + x_2 = 4 = x_4$. Case 11 is not possible because $x_1 + x_2 = 5 = x_4$. Using similar arguments, cases 1, 8, 10 and 12 can be eliminated. The only remaining cases are 4, 6, 7 and 9.

In case 4, $x_3 + x_5 = 16$. Thus, $x_3 \in \{8,9,...\}$ and $x_5 \in \{8,9,...\}$ which is impossible because $8 + 9 > 16$.

In case 7, $x_3 + x_5 = 15$. 8 $\{x_3,x_5\}$ because $x_6 + x_7 = 8$. This implies that $x_3 \in \{7,9,...\}$ and $x_5 \in \{7,9,...\}$ which is impossible because $7 + 9 > 15$. In case 6, $x_3 + x_5 = 15$, which implies $\{x_3,x_5\} = \{6,9\}$. In case 9, $x_3 + x_5 = 14$ and this gives $\{x_3,x_5\} = \{6,8\}$. Using each of the two possible values for x_3 and x_5, the first four rows of a sum triangle are obtained. However, none of them yields a solution. Thus, $\tau(7) \geq 33$. Using methods similar to the above as well as a computer program requiring several seconds of CPU time, $\tau(7) \neq 33$ was obtained. Table 1 lists a sum triangle with $\tau(7) = 34$, also obtained by computer.

Proof that $\tau(8) = 44$:

Consider the sum

$$S_1 + S_2 + S_3 + s_{14} + s_{58}$$
$$= 7\tau(8) - 3(x_1+x_8) - (x_2+x_7) \geq \sigma(23) = 276. \qquad (8)$$

Thus, $7\tau(8) \geq 276 + 3(x_1 + x_8) + (x_2 + x_7)$. Using properties of the sum triangle, $3(x_1+x_8) + (x_2+x_7) \geq 17$. This yields $7\tau(8) \geq 293$, or $\tau(8) \geq 42$. Suppose $\tau(8) = 42$. Using (8),

$$294 \geq 276 + 3(x_1+x_8) + (x_2+x_7),$$

which gives $3(x_1+x_8) + (x_2+x_7) \leq 18$. let $\xi = 3(x_1+x_8) + (x_2+x_7)$. Then $\xi \in \{17,18\}$.

Consider the case $\xi = 17$. Assume that $x_1 = 1$ and $x_8 = 2$. Then $x_2 = 3$ and $x_7 = 5$. This yields $\{x_3,x_4,x_5,x_6\} \subset \{6,8,9,10,...\}$. However, $x_1 + x_2 + x_7 + x_8 + 33 = 44$, which contradicts the assumption $\tau(8) = 42$.

Now let $\xi = 18$. If $(x_1+x_8) \geq 4$, then $(x_2+x_7) \leq 6$. This yields $\{x_1,x_8\} = \{1,3\}$ and $\{x_2,x_7\} = \{2,4\}$; which leads to a contradiction. The above implies that $(x_1+x_8) = 3$ and $(x_2+x_7) = 18 - 9 = 9$. Thus $\{x_1,x_8\} = \{1,2\}$ and $\{x_2,x_7\} \subset \{3,6\} \cup \{4,5\}$. This gives rise to the following four cases:

Case	X_1	X_2	X_7	X_8
1	1	3	6	2
2	1	4	5	2
3	1	5	4	2
4	1	6	3	2

Case 2 is not possible because $x_1 + x_2 = 5 = x_7$. In case 3, $x + x_2 = 6 = x_7 + x_8$. Case 1 gives $x_3 + x_4 + x_5 + x_6 = 42 - 12 = 30$. Using properties of the sum triangle, $\{x_3,x_4,x_5,x_6\} \subset \{5,7,9,10,...\}$. Thus, $x_3 + x_4 + x_5 + x_6 \geq 31$. For case 4, $x_3 + x_4 + x_5 + x_6 = 30$ and $\{x_3,x_4,x_5,x_6\} \subset \{4,8,9,10,...\}$. As before, $x_3 + x_4 + x_5 + x_6 = 31$. One concludes that $\tau(8) \geq 43$.

Applying the same techniques as above it can be shown that

$$\tau(8) \neq 43.$$

Table 1 lists a sum triangle with $\tau(8) = 44$.

Proof that $\tau(9) = 55$:

Consider the sum

$$S_1 + S_2 + s_{13} + s_{46} + s_{79}$$
$$= 4\tau(9) - (x_1+x_9) \geq \sigma(20) = 210.$$

This yields $4\tau(9) \geq 210 + (x_1+x_9) = 213$, or $\tau(9) \geq 54$. Table 1 lists a sum triangle with $\tau(9) = 55$.

To complete the proof it remains to show that $\tau(9) = 54$ is not possible. Using a computer program similar to that for $\tau(8)$, it was found that $\tau(9) \neq 54$ after several minutes of CPU time.

For $n = 10, 11, 12$ and 13 the above methods yielded the following inequalities:

$$\tau(10) \geq 67, \quad \tau(11) \geq 81, \quad \tau(12) \geq 98 \quad \text{and} \quad \tau(13) \geq 116.$$

For $n = 10$, consider the sum

$$S_1 + S_2 + s_{13} + s_{24} + s_{46}$$
$$= S_1 + S_2 + S_3 - s_{25} - s_{68}$$
$$= 5\tau(10) - 2(x_1+x_{10}) \geq \sigma(25) = 325.$$

Thus, $5\tau(10) \geq 325 + 2(x_1+x_{10}) \geq 331$, which yields $\tau(10) \geq 67$. For $n = 11$, consider the sum

$$S_1 + S_2 + S_3 = 6\tau(11) - 3(x_1+x_{11}) - (x_2+x_{10})$$
$$\geq \sigma(30) = 465.$$

Thus, $6\tau(11) \geq 465 + 3(x_1+x_{11}) + (x_2+x_{10})$. Since

$$3(x_1+x_{11}) + (x_2+x_{10}) \geq 17, \quad 6\tau(11) \geq 482.$$

This gives $\tau(11) \geq 81$.

For $n = 12$, consider the sum

$$S_1 + S_2 + S_3 + s_{14} + s_{58} + s_{9,12}$$
$$= 7\tau(12) - 3(x_1+x_{12}) - (x_2+x_{11}) \geq \sigma(36) = 666.$$

Thus,

$$7\tau(12) \geq 666 + 3(x_1+x_{12}) + (x_2+x_{11})$$
$$\geq 666 + 9 + 8 \geq 683.$$

This gives $\tau(12) \geq 98$.

For $n = 13$, consider the sum

$$S_1 + S_2 + S_3 + s_{14} + s_{25} + s_{58} + s_{69} + s_{9,12} + s_{10,13}$$
$$= 3\tau(13) - 4(x_1 + x_{13}) - (x_1+x_{12}) \geq \sigma(42) = 903.$$

Thus,

$$3\tau(13) \geq 903 + 4(x_1+x_{13}) + (x_1+x_{12})$$
$$\geq 903 + 12 + 8 \geq 923.$$

This gives $\tau(13) \geq 116$.

Using the estimates $t_3(10)$, $t_3(11)$, $t_3(12)$ and $t_3(13)$ listed in Table 2 as well as a computer program requiring several hours of CPU time, the results $\tau(10) = 72$, $\tau(11) = 85$, $\tau(12) = 106$ and $\tau(13) = 127$ were obtained.

The effectiveness of the methods used in finding $\tau(n)$ for the values of n considered is exhibited in Table 2. For $n < 7$, the values of $\tau(n)$ were calculated directly. For $n \in \{7,8,9\}$ the estimates of $\tau(n)$, namely t_3, were close to the solutions. A computer program requiring several minutes of CPU time yielded the exact values.

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