A3037
A5208
A117618
A117632
A259466

[BS41]

# THE NOTION OF COMPLEXITY

by

W. A. Beyer, M. L. Stein, and S. M. Ulam

## ABSTRACT

The notion of the arithmetic complexity $|n|$ of an integer n is defined in terms of the minimum number of additions, multiplications, and exponentiations required to combine 1's to form n. The value of $|n|$ is calculated for $n < 2^{10}$. n is called complicated if $|n| > |n_1|$ for every $n_1 < n$. Of the first 19 complicated numbers, 14 are prime. A conjecture about a relation between complexity and entropy is proposed. Some computations are presented to support this conjecture.

---

## I. INTRODUCTION

In this report we discuss notions of complexity in some algebraic structures. These notions are also applicable to more general combinatorial situations that perhaps lack any algebraic pattern in the classical sense. We concentrate on a few special cases for which we studied and calculated a special notion of complexity. Essentially, we examined a special notion of complexity for ordinary integers with a little excursion on such a notion for integers <u>modulo a prime</u>.

The notion of complexity, in our view, is separate, though associated with the idea of the amount of <u>information</u> or <u>entropy</u> of a system. We mention briefly a possible axiomatic approach to defining a real number called complexity for elements of a set or of a class on which certain <u>operations</u> are performed. These could be binary operations; our set could be a set of integers, and the operations could be addition, multiplication, and exponentiation, for example. It is this case that was examined on a computing machine and to which most of this report is devoted.

Another case would be a class of subsets of a given set, with allowed operations being the Boolean operations of <u>union</u> and <u>intersection</u> or <u>union</u> and <u>complementation</u>. One could add other operations, for example, the direct product of sets and also <u>projection</u>. This would correspond to allowing quantifiers in our theory. One can study a notion of complexity for vectors in a countable space or even in the continuum. An important study would be that of a <u>relative</u> complexity; that is to say, complexity of elements or "expressions" when the complexity of certain symbols is normalized to 1. In what has been sometimes called "speculation" on constants in physical theories, for example, the whole art seems to depend on the success of attempts to define some known important numbers, e.g., the dimensionless ratios

$$M_{proton}/M_{electron} = 1836.11...$$

and

$$e^2/hc = 137.1...$$

by use of only a few artificially introduced constants which should be as "simple" as possible. (cf. the attempts by Eddington[1] and some very recent ones by Good[2] and Wyler.[3])

Considered "genetically," a mathematical theory resembles a tree in that one obtains from a given number of symbols corresponding to "variables" and from a number of allowed operations, expressions that elongate by branching. The simplifications and abbreviations may then reduce the length of the expressions.

One could try to define complexity in a mathematical structure by postulating certain of its properties, somewhat like postulating properties of a measure.

Let the structure, S, consist of elements x, y, .... It may be finite or infinite. We have in the set S a number of, say, binary operations $R_1$, $R_2$, ... $R_n$. We want to assign a number $c(x) \geqslant 0$ to each element x of S and to each $R_i$ $(i = 1 \ldots n)$ so that the following properties should hold.

a. If $z = R_i(x,y)$, then $c(z) = c(R_i(x,y)) \leqslant c(x) + c(y) + c(R_i) \qquad i = 1 \ldots n.$

b. For each element z, if $z = R_j(x,y)$, we should have for one case at least, $c(z) = c(x) + c(y) + c(R_j)$.

c. $c(x_0) = c(x_1) = \ldots c(x_n) = 1$ for some preassigned elements $x_0 \ldots x_n$ in S.

Needless to say, one can define analogous desiderata for the case in which the operations are more general than binary ones.

Obviously, in the case to which our exercise is devoted, these postulates are satisfied. Moreover, they define the complexity uniquely if, as must be the case in general, the complexity was normalized for some elements. (In our case, we assume the complexity of the integer 1 to be equal to 0.) We hope to study this notion more thoroughly for the more general case and also to perform experiments to determine complexity functions for the case in which S is a class of sets.) Ultimately, one would wish to discuss the complexity of genetic codes and biological organisms quantitatively.

("Integer" always means a positive integer.)

## II. ARITHMETIC COMPLEXITY OF INTEGERS

The arithmetic complexity $|n|$ of an integer n is defined as the fewest number of operators: +, x, xx (addition, multiplication, and exponentiation) which combine 1's to form n. Thus, $|1| = 0$; $|2| = 1$ since $2 = 1 + 1$; and $|5| = 4$ since $5 = (1 + 1)\text{xx}(1 + 1) + 1$ and not fewer than four operators with 1's will form five. Obviously, for a and b integers, $|a + b|$, $|ab|$, and $|a^b|$ are each not more than $|a| + |b| + 1$. For an infinity of integers n, the relation $|n + 1| = |n| + 1$ holds.

For the purpose of calculating the complexity of some integers, all correct formulas (up to some number of operators) involving +, x, xx, and the number 1 were enumerated using parenthesis-free notation on a computer. It required one hour of computer time to enumerate the integers with complexity $\leqslant 6$. Ralph Cooper made the following observation. Each correct formula involving n ($> 0$) operators is the composition of two formulas, one formula with $n_1$ operators and one formula with $n_2$ operators such that $n = n_1 + n_2 + 1$. One generates the integers of complexity n by first generating tables of integers of complexity $< n$. One partitions $n - 1$ into $n_1 + n_2$ in all ways and combines the integers of complexity $n_1$ with the integers of complexity $n_2$ to produce integers of complexity not larger than n. This method is considerably more efficient than the previous method. Table I lists the complexity of all integers $< 2^{10}$.

From the above construction, one sees that an upper bound $\ell^1(k)$ to $\ell(k)$, the number of integers of complexity k, is given by the solution of

$$\ell^1(k + 1) = \sum_{j=0}^{k} \ell^1(j)\, \ell^1(k - j),$$

with $\ell^1(0) = 1$. The solution to this equation is given by

$$\ell^1(k) = \frac{1}{k+1}\binom{2k}{k} 2^{-k},$$

which implies that

$$\ell(k) \leqslant \frac{2^k}{k\sqrt{\pi k}} + O\left(2^k\, k^{-5/2}\right).$$

Two additional forms of complexity have been considered and calculated.

a. Complement complexity. To make complexity symmetric in 0's and 1's, we introduce a slightly different complexity, the complement complexity $\overline{K}(y|n)$. Define the complement operation C by $C(x|n) = 2^n - 1 - x$. $\overline{K}(y|n)$ is defined as the fewest operations of addition, multiplication, exponentiation, and complementation that combine 1's to form y. In the count of operations, the

A3037

TABLE I. COMPLEXITY OF INTEGERS < $2^{10}$.

| Complexity | Integer |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 3 |
| 3 | 4 |
| 4 | 5 6 8 9 |
| 5 | 7 10 16 27 |
| 6 | 11 12 17 18 25 28 32 36 64 81 256 512 |
| 7 | 13 14 15 19 20 24 26 29 33 37 49 54 65 82 100 125 128 216 243 257 513 729 1024 |
| 8 | 21 22 30 34 38 48 50 55 56 66 72 83 101 121 126 129 144 162 217 244 258 289 324 343 514 625 730 784 1000 |
| 9 | 23 31 35 39 40 45 51 52 57 58 67 73 74 75 84 96 98 102 108 122 127 130 145 163 164 169 192 196 200 218 225 245 250 259 290 325 344 361 400 432 486 515 576 626 676 731 768 785 841 1001 |
| 10 | 41 42 44 46 53 59 60 63 68 76 78 80 85 87 90 97 99 103 109 110 111 112 123 131 132 135 146 147 165 166 170 193 195 197 201 202 219 226 242 246 251 252 260 288 291 300 326 345 362 375 384 401 433 434 441 484 487 488 516 577 578 627 648 677 686 732 769 771 786 842 900 1002 |
| 11 | 43 47 61 62 69 70 77 79 86 88 89 91 104 113 114 116 124 133 134 136 140 148 150 153 160 167 168 171 180 189 194 198 203 204 220 224 227 247 249 253 254 261 262 264 265 270 292 301 303 320 327 328 338 346 363 376 378 385 387 392 402 405 435 436 442 450 465 489 490 500 517 518 520 521 529 579 580 628 649 650 651 678 687 688 722 733 770 772 774 787 800 843 864 867 901 961 972 1003 |
| 12 | 71 92 93 95 105 106 115 117 118 119 120 137 141 149 151 152 154 156 161 172 174 175 176 181 185 190 199 205 206 208 221 222 228 232 234 248 255 263 266 271 272 280 283 293 294 296 297 302 304 306 321 329 330 332 333 339 340 347 360 364 366 377 379 381 386 388 390 393 394 403 404 406 410 437 438 443 448 451 452 459 491 492 501 502 504 507 519 522 528 530 539 567 581 582 585 588 600 629 640 652 654 656 675 679 689 690 723 724 734 735 737 738 750 756 773 775 777 788 801 802 810 844 865 866 868 870 882 902 962 968 973 974 975 976 1004 |
| 13 | 94 107 138 142 155 157 158 159 173 177 178 182 186 187 191 207 209 223 229 231 233 235 240 267 268 273 274 275 281 284 295 298 305 307 308 309 322 331 334 336 337 341 342 348 349 351 352 365 367 369 370 380 382 389 391 395 396 407 408 411 415 416 425 439 440 444 449 453 454 455 460 464 476 493 494 495 498 503 505 506 508 510 523 524 531 537 540 544 548 568 574 583 584 586 589 591 592 593 594 601 602 603 605 606 612 630 631 633 634 641 645 653 655 657 664 680 691 692 700 702 704 720 725 726 736 739 745 747 751 752 753 757 776 778 780 783 789 790 792 793 803 804 808 811 820 845 869 871 872 873 875 883 884 891 896 903 909 963 969 970 977 978 980 999 1005 1006 1008 1009 |
| 14 | 139 143 179 183 184 188 210 212 230 236 237 238 241 269 276 279 282 285 286 299 310 312 315 316 319 323 335 350 353 356 359 368 371 372 383 397 398 399 409 412 417 420 426 445 456 461 462 465 468 472 475 477 480 496 499 509 511 525 526 527 532 536 538 541 542 545 549 550 560 561 566 569 575 587 590 595 604 607 608 609 610 613 632 635 637 642 646 658 660 665 666 672 681 682 684 685 693 694 701 703 705 707 715 721 727 728 740 741 746 748 754 755 758 759 761 762 765 779 781 791 794 795 805 806 809 812 815 816 821 825 830 832 833 846 847 849 850 874 876 880 885 886 892 897 904 910 918 924 925 928 936 960 964 971 979 981 982 984 985 1007 1010 1014 1016 |
| 15 | 211 213 214 239 277 278 287 311 313 314 317 318 354 355 357 373 374 413 414 418 421 423 424 427 429 446 447 457 458 463 466 469 470 473 478 481 483 497 533 534 543 546 551 555 562 570 596 597 599 611 614 615 616 618 621 624 636 638 643 644 647 659 661 662 663 667 668 670 673 674 683 695 696 698 706 708 714 716 742 743 744 749 760 763 764 766 782 796 798 807 813 814 817 822 824 826 829 831 834 836 837 840 848 851 854 855 857 858 877 878 879 881 887 888 889 893 898 905 906 908 911 912 913 919 920 926 927 929 931 935 937 945 950 952 957 965 983 986 987 988 990 996 1011 1012 1015 1017 1018 1020 |
| 16 | 215 358 419 422 428 430 467 471 474 479 482 535 547 552 556 557 558 559 563 564 565 571 572 573 598 617 619 620 622 639 669 671 697 699 709 711 712 713 717 718 767 797 799 818 819 823 827 828 835 838 852 853 856 859 861 890 894 899 907 914 915 916 917 921 922 930 932 938 944 946 951 953 954 958 966 967 989 991 992 993 997 998 1013 1019 1021 1022 1023 |
| 17 | 431 553 554 623 710 719 839 860 862 895 923 933 939 940 941 942 947 948 949 955 956 959 994 995 |

first three are given the value 1 and the last is given the value zero. Thus $\bar{K}(y|n) = \bar{K}(2^n - 1 - y|n)$. Table II gives the values of $\bar{K}(y|n)$ for $y < 2^{10}$ and $n = 10$.

b. Modulo a prime p complexity. In addition to the operations of +, x, and xx, the operation of $\text{mod}_p$ is allowed and is defined by $\text{mod}_p(x) = x - p[x/p]$ where p is a fixed prime and [] denotes the greatest integer. Table III gives the modulo prime p = 137 complexity for integers < 137. Table IV gives the modulo prime p = 1009 complexity for integers < 1009.

## III. COMPLICATED NUMBERS

One defines n to be a complicated number if $|n| > |n_1|$ for every $n_1 < n$. The complicated numbers $< 2^{10}$ are 1, 2, 3, 4, 5, 7, 11, 13, 21, 23, 41, 43, 71, 94, 139, 211, 215, 431, and 863. (Those underlined are also prime.) Obviously, there are an infinity of complicated numbers. We propose the following conjectures.

a. There exists K such that all complicated numbers $K_1 > K$ are prime.

b. Every sufficiently large integer n is the sum of $k < \log n$ complicated integers.

c. There exists c such that every sufficiently large n satisfies $|n| < c + \sqrt{\log n}$.

## IV. COMPLEXITY AND ENTROPY

Kolmogorov[4,5] has introduced the notion of complexity of a finite string over a given alphabet. For simplicity, suppose the alphabet to be {0,1}. Let A be an algorithm that transforms finite binary sequences into binary sequences. By an algorithm is meant any of the various equivalent concepts used in logic. For a binary string x, one defines the complexity by

$$K_A(x) = \begin{cases} \min\limits_{A(p)=x} \ell(p) \\ \infty \quad \text{if no p exists such that } A(p) = x, \end{cases}$$

where $\ell(p)$ denotes the length of the binary string p. Analogously, one defines conditional complexity. Let A(p,x) be an algorithm defined from pairs of binary strings to binary strings. Put

$$K_A(y|x) = \begin{cases} \min\limits_{A(p,x)=y} \ell(p) \\ \infty \quad \text{if no p exists such that } A(p,x) = y. \end{cases}$$

$K_A(y|x)$ is called the conditional complexity of y with respect to x. Kolmogorov regards complexity as analogous to entropy. We make the following conjecture.

Conjecture. Let a discrete binary information source S in the sense of Shannon[6] be given with entropy $H = -p \log p - (1-p) \log (1-p)$ where probability (0) = p and probability (1) = 1 - p; $0 < p < 1$. Let $\{x_1, x_2, \ldots, x_{2^n}\}$ be the set of all binary strings of length n arranged in order of decreasing probability. Let k(n) be the least integer so that $\sum_{i=1}^{k(n)} \text{prob}\,(x_i) > r$ where $1/2 < r < 1$. Then asymptotically for large n,

$$H \approx \frac{1}{k(n)} \sum_{i=1}^{k(n)} K_A(x_i|n). \tag{1}$$

(In Eq. (1), $K_A$ should be normalized so that when p = 1/2,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} K_A(x_i|n) = 1.)$$

In other words, the most likely sequences from A have complexity approximately equal to the entropy of S.

In order to test the conjecture expressed in Eq. (1), we replaced $K_A(x_i|n)$ by $\lambda\bar{K}(y|n)$, where $\lambda$ is selected so that when p = 1/2,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \lambda\bar{K}(x_i|n) = 1.$$

Graphs of $H_1 = -p \log p - (1-p) \log (1-p)$ and

$$H_2 = \frac{1}{k(n)} \sum_{i=1}^{n} \lambda\bar{K}(x_i|n)$$

when n = 10 and r = .75 are shown in Fig. 1

A259466

TABLE II. COMPLEMENT COMPLEXITY OF INTEGERS $< 2^{10}$.

| Complement Complexity | Integer |
|---|---|
| 0 | 0 1 1022 1023 |
| 1 | 2 1021 |
| 2 | 3 1020 |
| 3 | 4 1019 |
| 4 | 5 6 8 9 1014 1015 1017 1018 |
| 5 | 7 10 16 27 996 1007 1013 1016 |
| 6 | 11 12 15 17 18 25 26 28 32 36 64 81 256 511 512 767 942 959 987 991 995 997 998 1005 1006 1008 1011 1012 |
| 7 | 13 14 19 20 24 29 31 33 35 37 49 54 63 65 80 82 100 125 128 216 243 255 257 294 510 513 729 766 768 780 807 895 898 923 941 943 958 960 969 974 986 988 990 992 994 999 1003 1004 1009 1010 |
| 8 | 21 22 23 30 34 38 48 50 52 53 55 56 62 66 72 79 83 99 101 121 124 126 127 129 144 162 215 217 225 239 242 244 254 258 289 293 295 324 343 347 398 509 514 625 676 680 699 728 730 734 765 769 779 781 784 798 806 808 861 879 894 896 897 899 902 922 924 940 944 951 957 961 967 968 970 971 973 975 985 989 993 1000 1001 1002 |
| 9 | 39 40 45 47 51 57 58 61 67 70 71 73 74 75 78 84 96 98 102 108 120 122 123 130 143 145 160 161 163 164 169 182 192 196 200 214 218 224 226 238 240 241 245 250 253 259 288 290 292 296 323 325 342 344 346 348 361 397 399 400 432 435 447 486 508 515 537 576 588 591 623 624 626 662 675 677 679 681 698 700 727 731 733 735 764 770 773 778 782 783 785 797 799 805 809 823 827 831 841 854 859 860 862 863 878 880 893 900 901 903 915 921 925 927 939 945 948 949 950 952 953 956 962 965 966 972 976 978 983 984 |
| 10 | 41 42 44 46 59 60 68 69 76 77 85 87 90 93 95 97 103 104 105 106 107 109 110 111 112 119 131 132 135 141 142 146 147 158 159 165 166 168 170 181 183 189 191 193 195 197 198 199 201 202 213 219 223 227 237 246 248 249 251 252 260 287 291 297 300 322 326 329 337 341 345 349 360 362 375 384 396 401 430 431 433 434 436 437 441 445 446 448 450 478 484 485 487 488 494 507 516 529 535 536 538 539 545 573 575 577 578 582 586 587 589 590 592 593 622 627 639 648 661 663 674 678 682 686 694 697 701 723 726 732 736 763 771 772 774 775 777 786 796 800 804 810 821 822 824 825 826 828 830 832 834 840 842 853 855 857 858 864 865 876 877 881 882 888 891 892 904 911 912 913 914 916 917 918 919 920 926 928 930 933 936 938 946 947 954 955 963 964 977 979 981 982 |
| 11 | 43 86 88 89 91 92 94 113 114 116 118 133 134 136 138 140 148 150 153 156 157 167 171 180 184 186 188 190 194 203 204 208 212 220 222 228 229 234 236 247 261 262 264 265 270 286 298 299 301 303 306 320 321 327 328 330 331 335 336 338 339 340 350 359 363 364 372 373 374 376 377 378 381 383 385 387 392 395 402 405 428 429 438 439 440 442 443 444 449 451 452 476 477 479 480 482 483 489 490 493 495 500 502 503 504 505 506 517 518 519 520 521 523 528 530 533 534 540 541 543 544 546 547 571 572 574 579 580 581 583 584 585 594 595 618 621 628 631 636 638 640 642 645 646 647 649 650 651 659 660 664 673 683 684 685 687 688 692 693 695 696 702 703 717 720 722 724 725 737 753 758 759 761 762 776 787 789 794 795 801 803 811 815 819 820 829 833 835 837 839 843 852 856 866 867 870 873 875 883 885 887 889 890 905 907 909 910 929 931 932 934 935 937 980 |
| 12 | 115 117 137 139 149 151 152 154 155 172 174 175 176 179 185 187 205 206 207 209 210 211 221 230 231 232 233 235 263 266 267 269 271 272 273 279 280 282 283 284 285 302 304 305 307 309 315 316 318 319 332 333 334 351 358 365 366 367 369 371 379 380 382 386 388 390 391 393 394 403 404 406 410 416 423 426 427 453 454 456 459 474 475 481 491 492 496 498 499 501 522 524 525 527 531 532 542 548 549 564 567 569 570 596 597 600 607 613 617 619 620 629 630 632 633 635 637 641 643 644 652 654 656 657 658 665 672 689 690 691 704 705 707 708 714 716 718 719 721 738 739 740 741 743 744 750 751 752 754 756 757 760 788 790 791 792 793 802 812 813 814 816 817 818 836 838 844 847 848 849 851 868 869 871 872 874 884 886 906 908 |
| 13 | 173 177 178 268 274 275 276 277 278 281 308 310 312 314 317 352 353 354 355 356 357 368 370 389 407 408 409 411 415 417 418 420 421 422 424 425 455 457 458 460 463 464 465 468 472 473 497 526 550 551 555 558 559 560 563 565 566 568 598 599 601 602 603 605 606 608 612 614 615 616 634 653 655 666 667 668 669 670 671 706 709 711 713 715 742 745 746 747 748 749 755 845 846 850 |
| 14 | 311 313 412 413 414 419 461 462 466 467 469 470 471 552 553 554 556 557 561 562 604 609 610 611 710 712 |

TABLE III. MODULO PRIME $p = 137$ COMPLEXITY OF INTEGERS < 137.

| Complexity | Integer | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | | | | | | |
| 2 | 3 | | | | | | | | | | | | | | | | | | | |
| 3 | 4 | | | | | | | | | | | | | | | | | | | |
| 4 | 5 | 6 | 8 | 9 | | | | | | | | | | | | | | | | |
| 5 | 7 | 10 | 16 | 27 | | | | | | | | | | | | | | | | |
| 6 | 11 | 12 | 17 | 18 | 25 | 28 | 32 | 36 | 64 | 81 | 101 | 119 | | | | | | | | |
| 7 | 13 | 14 | 15 | 19 | 20 | 24 | 26 | 29 | 33 | 37 | 44 | 49 | 50 | 54 | 61 | 65 | 79 | 82 | 92 | 100 |
| | 102 | 106 | 120 | 122 | 125 | 128 | | | | | | | | | | | | | | |
| 8 | 21 | 22 | 30 | 34 | 38 | 41 | 45 | 48 | 51 | 55 | 56 | 60 | 62 | 63 | 66 | 68 | 69 | 72 | 77 | 80 |
| | 83 | 88 | 93 | 99 | 103 | 107 | 109 | 117 | 118 | 121 | 123 | 126 | 129 | 130 | 132 | 133 | | | | |
| 9 | 23 | 31 | 35 | 39 | 40 | 42 | 46 | 47 | 52 | 53 | 57 | 58 | 59 | 67 | 70 | 73 | 74 | 75 | 76 | 78 |
| | 84 | 87 | 89 | 94 | 96 | 98 | 104 | 108 | 110 | 111 | 112 | 113 | 115 | 124 | 127 | 131 | 134 | 136 | | |
| 10 | 0 | 43 | 71 | 85 | 86 | 90 | 95 | 97 | 105 | 114 | 116 | 135 | | | | | | | | |
| 11 | 91 | | | | | | | | | | | | | | | | | | | |

## V. COMPLEXITY OF N-TUPLES OF INTEGERS

Matijasevič[7] has proved the following theorem. There exists a fifth-degree polynomial $Q(y_1, \ldots, y_k; z)$ with integer coefficients such that any enumerable set m of natural numbers (for example, the set of prime numbers) coincides with the set of natural values of the polynomial $Q(y_1, \ldots, y_k; a_m)$ where $a_m$ is a certain number effectively constructed for the set m. From the result, it follows that if one could discuss complexity of n-tuples of integers, then one could discuss the complexity of enumerable sets of natural numbers by equating such complexity to the complexity of the associated polynomial Q.

## REFERENCES


1. A. S. Eddington, Fundamental Theory (Cambridge University Press, 1946).
2. I. J. Good, "The Proton and Neutron Masses and a Conjecture for the Gravitational Constant," Phys. Let. 33A, 383-384 (1970).
3. A. Wyler, "Les Groupes des Potentiels de Coulomb et de Yukawa," Compt. Rend. Acad. Sci. Paris, 271, 186-188 (1971).
4. A. Kolmogorov, "Three Approaches for Defining the Concept of Information Quantity," Problems of Information Transmission 1, 3-11 (1965).
5. A. Kolmogorov, "Logical Basis for Information Theory and Probability Theory," IEEE Trans. Information Theory IT-14, 662-664 (1968).
6. C. E. Shannon and Warren Weaver, The Mathematical Theory of Communication (The University of Illinois Press, Urbana, 1949).
7. Ju V. Matijasevič, "Enumerable Sets are Diophantine," Soviet Math. Dokl. 11, 354-358 (1970).


## ADDITIONAL REFERENCES TO COMPLEXITY NOT USED IN TEXT


1. E. L. Lawler, "The Complexity of Combinatorial Computations: A Survey," Proceedings of 1971 Polytechnic Institute of Brooklyn Symposium on Computers and Automata.
2. D. W. Loveland, "On Minimal Program Complexity Measures," ACM Symp. Theory of Computing, Marina del Rey, California, May 5-7, 1969.
3. P. Young, "A Note on Dense and Nondense Families of Complexity Classes," Math. Systems Theory 5, 66-70 (1971).

TABLE IV. <u>MODULO PRIME</u> p = 1009 COMPLEXITY OF INTEGERS < 1009.

| Complexity | Integer |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 3 |
| 3 | 4 |
| 4 | 5 6 8 9 |
| 5 | 7 10 16 27 |
| 6 | 11 12 17 18 25 28 32 36 64 81 256 512 |
| 7 | 13 14 15 19 20 24 26 29 33 37 49 54 65 82 100 125 128 216 243 257<br>507 513 548 729 960 |
| 8 | 21 22 30 34 38 48 50 55 56 60 66 72 74 83 87 101 121 126 129 137<br>144 162 169 217 244 258 287 289 324 343 383 384 508 514 527 549 625 710 730 763<br>784 813 911 961 993 1000 |
| 9 | 23 31 35 39 40 45 51 52 57 58 61 67 73 75 80 84 88 96 98 102<br>106 120 122 127 130 138 142 145 148 163 164 170 173 174 189 192 196 200 218 225<br>240 242 245 250 259 270 271 274 288 290 322 325 344 360 361 385 400 411 432 449<br>464 480 486 490 509 515 528 538 550 572 573 576 617 626 631 635 640 670 676 707<br>711 713 719 731 764 766 768 782 785 787 808 814 829 841 859 877 893 898 912 919<br>926 962 977 985 994 1001 |
| 10 | 41 42 44 46 53 59 62 63 68 76 78 85 89 90 97 99 103 105 109 110<br>111 112 123 131 132 135 139 143 146 147 149 165 166 171 175 177 179 180 185 186<br>190 193 195 197 201 202 203 219 222 226 241 246 251 252 253 254 260 272 275 280<br>284 286 291 296 300 309 320 323 326 328 338 345 348 362 375 386 394 401 404 412<br>421 423 431 433 434 435 441 443 450 451 454 465 481 482 484 487 488 491 497 506<br>510 516 517 519 523 529 530 539 540 551 555 556 559 574 577 578 605 607 609 616<br>622 627 632 636 641 648 663 671 675 677 686 708 709 712 714 715 720 726 732 741<br>755 765 767 769 771 777 783 786 788 791 805 809 815 822 824 830 835 842 847 860<br>861 862 878 881 882 886 894 896 899 900 906 913 920 922 927 929 935 937 942 945<br>955 963 972 978 979 986 991 995 999 1002 1006 1007 |
| 11 | 43 47 69 70 71 77 79 86 91 104 106 113 114 116 124 133 134 136 140 150<br>153 160 167 168 172 176 178 181 187 191 194 198 199 204 205 206 209 210 211 212<br>213 220 223 224 227 247 249 255 261 262 264 265 268 269 273 276 281 283 285 292<br>297 301 302 303 310 313 314 321 327 329 331 332 334 335 336 337 339 340 346 349<br>353 355 363 374 376 378 379 382 387 392 395 398 402 405 406 409 410 413 417 418<br>422 424 429 436 442 444 448 452 453 455 456 466 479 483 485 489 492 494 498 500<br>501 511 518 520 521 524 531 533 541 542 545 546 552 557 558 560 561 565 568 575<br>579 580 583 591 592 597 599 600 606 608 610 611 615 619 620 623 628 633 634 637<br>638 642 644 649 650 651 654 656 661 662 664 666 668 672 673 674 678 681 685 687<br>688 689 692 693 694 696 706 716 721 722 727 733 735 738 742 745 748 753 756 758<br>759 762 770 772 774 778 789 792 793 800 803 804 806 810 816 823 825 831 836 840<br>843 845 848 852 863 864 865 866 867 870 875 879 883 884 887 895 897 901 904 907<br>908 914 915 921 923 930 934 936 938 940 943 946 949 951 953 956 959 964 967 968<br>973 980 981 982 987 989 992 996 1003 1008 |
| 12 | 0 92 93 95 107 115 117 118 119 141 151 152 154 156 157 161 182 183 188 207<br>208 214 221 228 232 234 235 248 263 266 277 278 282 293 294 295 298 299 304 306<br>311 315 317 319 330 333 341 342 347 350 354 356 357 358 364 366 369 370 372 377<br>380 381 388 390 393 396 397 399 403 407 414 415 419 420 425 426 430 437 438 445<br>457 459 460 461 463 467 469 471 472 473 493 495 499 502 503 504 505 522 525 532<br>534 536 543 544 547 553 554 562 563 566 567 569 571 581 582 584 585 588 593 598<br>601 603 604 612 613 616 621 624 629 630 639 643 645 646 652 655 657 665 667 669<br>677 682 690 695 697 700 717 718 723 724 725 728 734 736 737 739 743 746 747 749<br>750 754 757 760 773 775 779 790 794 797 799 801 802 807 811 817 818 820 826 832<br>837 844 846 849 853 858 868 869 871 872 876 880 885 888 889 902 905 909 916 917<br>924 925 931 939 941 944 947 950 952 954 957 965 966 969 974 975 976 983 988 990<br>997 1004 1005 |
| 13 | 94 155 158 159 184 215 229 231 233 236 237 267 279 305 307 308 312 316 318 351<br>352 359 365 367 368 371 373 389 391 408 416 427 428 439 440 446 447 458 462 468<br>470 474 475 476 496 526 535 537 564 570 586 587 589 590 594 602 614 647 653 658<br>659 680 683 691 698 701 702 704 740 744 751 752 761 776 780 781 795 796 798 812<br>819 821 827 833 834 838 850 851 854 857 873 874 890 891 903 910 918 926 932 948<br>958 970 984 998 |
| 14 | 230 238 239 477 478 595 596 660 684 699 703 705 828 839 855 892 933 971 |
| 15 | 856 |

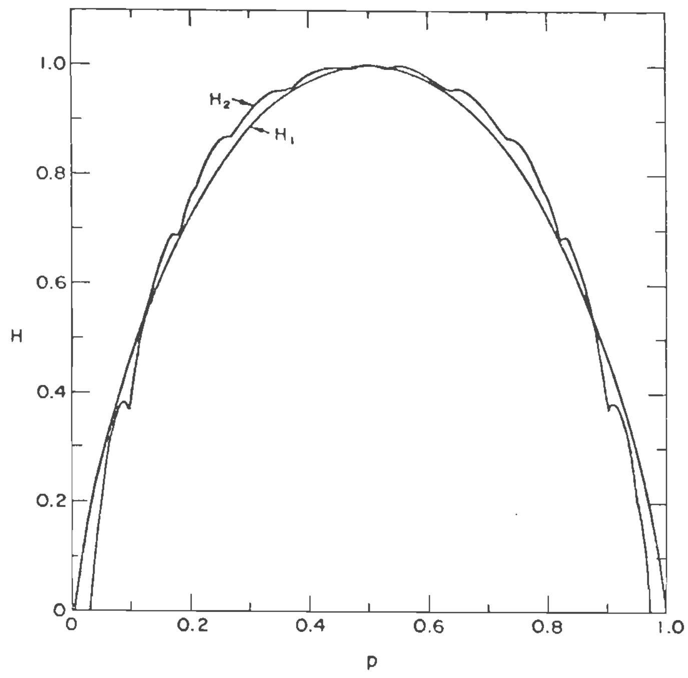


Fig. 1. Comparison of entropy $H_1 = -\sum p_i \log p_i$ and complement complexity $H_2$ as defined and discussed in text.

KT/bk: 304 (50)