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ENUMERATION OF EULER GRAPHS

Robert W. Robinson

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

INTRODUCTION

An Euler graph is a graph in which every point has even degree. The number of Euler graphs on a given number of points is treated as an application of Burnside's lemma. The number of connected Euler graphs is recovered by using Riddell's formula. The enumeration of self-complementary Euler graphs is reduced to the enumeration of ordinary self-complementary graphs by direct correspondence.

1. EULER GRAPHS ON n POINTS

Let H be a finite permutation group. Let $|H|$ denote the order of H and N_H the number of orbits of transitivity determined by H. If $h \in H$, let $j_m(h)$ be the number of cycles of length m in the disjoint cycle decomposition of h. Then according to Burnside's lemma [1],

$$N_H = \frac{1}{|H|} \sum_{h \in H} j_1(h).$$

By a *labeled* Euler graph on n points we mean one with point set $\{1, \ldots, n\}$. Let S_n be the group of all permutations on $\{1, \ldots, n\}$. For every $g \in S_n$ there is a corresponding permutation g^* on n-point labeled Euler graphs which is obtained when the points are relabeled according to g. Let $S_n{}^*$ be the homomorphic image of S_n under $*$. The number of orbits of $S_n{}^*$ is the number u_n of nonisomorphic (*unlabeled*) Euler graphs on n points. That is, every Euler graph on n points is isomorphic to one on $\{1, \ldots, n\}$ as point set, and two of these are isomorphic if and only if a permutation of $S_n{}^*$ maps one to the other.

In order to apply Burnside's lemma to the determination of

$$u_n = N_{S_n{}^*},$$

we need to know $j_1(g^*)$ for all $g \in S_n$. It is convenient in this connection to think of g as inducing a permutation $g^{(2)}$ on unordered pairs, or "lines," from $\{1, \ldots, n\}$. A labeled graph G is left invariant by g if and only if, for every cycle in the disjoint cycle decomposition of $g^{(2)}$, all or none of the lines of the cycle belong to G. Thus the total number of labeled graphs left invariant by g is $2^{\nu(g)}$ where $\nu(g) = \sum_{i=1}^{n} j_i(g^{(2)})$.

The *cycle type* of $g \in S_n$ is the ordered n-tuple $\langle j_1(g), j_2(g), \ldots, j_n(g)\rangle$ The number $\nu(g)$ is an invariant of the cycle type of g, so we let ν be defined on cycle types; thus

$$\nu(\langle j_1(g), \ldots, j_n(g)\rangle) = \nu(g).$$

To find $\nu(g)$ we consider separately the lines which join distinct point cycles of g and those which join the points of some single cycle of g. The ij lines which join distinct point cycles of lengths i and j are permuted by $g^{(2)}$ in (i, j) cycles of length $[i, j]$ each. Here (i, j) and $[i, j]$ are the greatest common divisor and least common multiple, respectively, of i and j. The lines from a single cycle of length $2i + 1$ fall into i cycles of length $2i + 1$, while the lines from a cycle of length $2i$ fall into $i - 1$ cycles of length $2i$ and one, the diagonal cycle, of length i. Thus

$$\nu(\langle \sigma_1, \ldots, \sigma_n\rangle) = \sum_{i<j} \sigma_i \sigma_j (i, j) + \sum_i i\left(\sigma_{2i} + \sigma_{2i+1} + \binom{\sigma_i}{2}\right).$$

This result is given by Harary [2, Eq. (10)].

It remains to determine how many of the $2^{\nu(g)}$ graphs on $\{1, \ldots, n\}$ left invariant by g are Euler graphs. First consider the points on a cycle of even length induced by g. In a labeled graph left invariant by g they must all have the same degree. The diagonal line cycle contributes just one adjacency at each point of the cycle, so that by including or excluding the diagonal cycle as need be they can all be made to have even degree. Essentially, one degree of freedom is lost for each cycle of even length induced by g in passing from all labeled graphs left invariant by g to just those which are Euler.

If the total number of point cycles of g of odd length is $m > 0$, let b be some cycle of odd length and let A be a set consisting of exactly one line cycle joining b with each of the $m - 1$ other cycles of odd length. In constructing a graph left invariant by g, each of the line cycles which is not in A may be included or excluded at will from the line set of the graph. If c is a point cycle of g of odd length which is distinct from b, then any line cycle joining b

contributes an odd number to the degree of every point of b and c. Thus the condition that the points of c have even degree determines whether or not the unique line cycle of A which joins b and c must be included in the graph. If all of the points on other cycles of odd length have even degree, then the points of b must have even degree too since any graph has an even number of points of odd degree. In all, when $m > 0$ we lose $m - 1$ degrees of freedom in the choice of line cycles for a graph invariant under g by requiring that the points on odd cycles of g have even degree. When $m = 0$ the restriction is vacuous; in general $m - sg(m)$ degrees of freedom are lost, where $sg(m) = 1$ if $m > 0$ and $sg(0) = 0$.

If we include the effect of the Euler restriction on the even cycles too, we see that[1] if g has cycle type $\langle \sigma_1, \ldots, \sigma_n \rangle$ then

$$j_1(g^*) = 2^{\mu(\langle \sigma_1, \ldots, \sigma_n \rangle)}$$

where

$$\begin{aligned}\mu(\langle \sigma_1, \ldots, \sigma_n \rangle) &= \nu(\langle \sigma_1, \ldots, \sigma_n \rangle) - \sum_i \sigma_i + sg\left(\sum_i \sigma_{2i+1}\right) \\ &= \sum_{i<j} \sigma_i \sigma_j (i, j) + \sum_i i \binom{\sigma_i}{2} + \sum_i (i-1)(\sigma_{2i} + \sigma_{2i+1}) + sg\left(\sum_i \sigma_{2i+1}\right). \qquad (1)\end{aligned}$$

There are

$$\frac{n!}{\prod_i i^{\sigma_i} \sigma_i!}$$

permutations in S_n of cycle type $\langle \sigma_1, \ldots, \sigma_n \rangle$. According to Burnside's lemma, then, as it applies to S_n^*, we see that

$$u_n = \sum \frac{2^{\mu(\langle \sigma_1, \ldots, \sigma_n \rangle)}}{\prod_i i^{\sigma_i} \sigma_i!}, \qquad (2)$$

the sum being over all ordered n-tuples $\langle \sigma_1, \ldots, \sigma_n \rangle$ such that $n = \sum_i i\sigma_i$. Together Eqs. (1) and (2) provide a straightforward method of finding u_n, the number of nonisomorphic Euler graphs on n points. In Table I the partitions of 6 are listed with the numbers appropriate to evaluating u_6 from (2); we find

$$\begin{aligned}u_6 &= \frac{2^{10}}{720} + \frac{2^7}{48} + \frac{2^4}{18} + \frac{2^6}{16} + \frac{2^3}{8} + \frac{2^3}{6} + \frac{2^2}{5} + \frac{2^6}{48} + \frac{2^4}{18} + \frac{2^3}{8} + \frac{2^2}{6} \\ &= 16.\end{aligned}$$

[1] The author is indebted to James C. Owings, Jr., for suggesting a simplification of the proof.

TABLE I

THE PARTITIONS OF 6 AND NUMBERS RELEVANT TO THE COMPUTATION OF u_6

$\langle\sigma_1, \ldots, \sigma_6\rangle$	$\nu(\langle\sigma_1, \ldots, \sigma_6\rangle)$	$\mu(\langle\sigma_1, \ldots, \sigma_6\rangle)$	$\prod_{i=1}^{6} \sigma_i! i^{\sigma_i}$
$\langle 6, 0, 0, 0, 0, 0\rangle$	15	10	720
$\langle 4, 1, 0, 0, 0, 0\rangle$	11	7	48
$\langle 3, 0, 1, 0, 0, 0\rangle$	7	4	18
$\langle 2, 2, 0, 0, 0, 0\rangle$	9	6	16
$\langle 2, 0, 0, 1, 0, 0\rangle$	5	3	8
$\langle 1, 1, 1, 0, 0, 0\rangle$	5	3	6
$\langle 1, 0, 0, 0, 1, 0\rangle$	3	2	5
$\langle 0, 3, 0, 0, 0, 0\rangle$	9	6	48
$\langle 0, 1, 0, 1, 0, 0\rangle$	5	3	8
$\langle 0, 0, 2, 0, 0, 0\rangle$	5	4	18
$\langle 0, 0, 0, 0, 0, 1\rangle$	3	2	6

In Fig. 1 are pictured the 16 Euler graphs on 6 points. There are 34 unlabeled graphs on 5 points. This witnesses the failure of the unlabeled analog to the correspondence, mentioned below, between labeled graphs on n points and labeled Euler graphs on $n + 1$ points.

2. CONNECTED EULER GRAPHS

A graph has a circuit which contains every line exactly once only if it is a connected Euler graph. Thus a natural interest attaches to the number U_n of nonisomorphic connected Euler graphs on n points. It is convenient to deal with the associated generating functions. Let $U(x) = \sum_{n=1}^{\infty} U_n x^n$ and $u(x) = \sum_{n=1}^{\infty} u_n x^n$. In Section 1 we described the computation of the coefficients of $u(x)$; it is found that

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$$u(x) = x + x^2 + 2x^3 + 3x^4 + 7x^5 + 16x^6 + 54x^7 + 243x^8 + \cdots.$$

A standard application of Polyà's counting methods, due to Riddell and described by Harary [2, Eq. (33)], relates the generating function for the connected graphs of a given sort to the generating function for all graphs of the given sort. In our case this takes the form

$$1 + u(x) = \exp \sum_{n=1}^{\infty} \frac{1}{n} U(x^n).$$

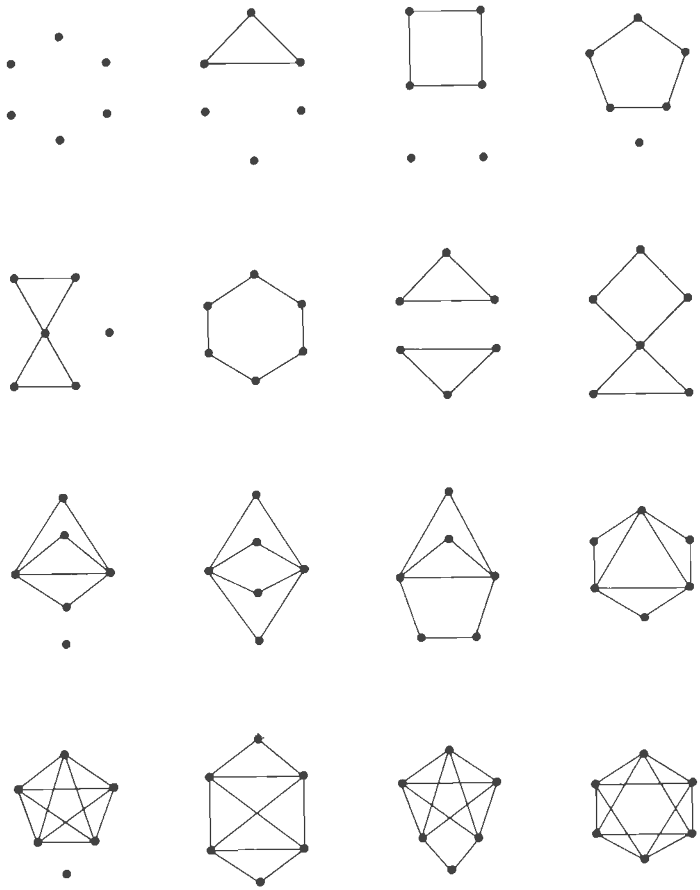

FIG. 1. The 16 Euler graphs on 6 points.

Taking logarithms and completing the inversion by use of the Möbius function μ, we find

$$U(x) = -\sum_{i=1}^{\infty} \frac{1}{i}\,\mu(i) \sum_{j=1}^{\infty} \frac{(-1)^j}{j}\, u(x^i)^j.$$

This relation applied to the first 8 coefficients of $u(x)$ gives

$$U(x) = x + x^3 + x^4 + 4x^5 + 8x^6 + 37x^7 + 184x^8 + \cdots.$$

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From Fig. 1 it can be verified that of the 16 Euler graphs on 6 points, 8 are connected.

3. Self-Complementary Euler Graphs

The *complement* $\bar{G}$ of a graph G has the same point set as G, and an unordered pair of points is a line in $\bar{G}$ just if it is not a line in G. A *self-complementary* graph is one which is isomorphic to its complement. A permutation on the points of G which is an isomorphism of G onto $\bar{G}$ is called an *inversion* of G. We show first that every self-complementary Euler graph has $4k + 1$ points for some k. Then we establish a one-to-one correspondence $G \leftrightarrow G^E$, between the self-complementary graphs on $4k$ points and the self-complementary Euler graphs on $4k + 1$ points. This correspondence is simply an unlabeled version of the correspondence between labeled graphs on n points and labeled Euler graphs on $n + 1$ points which is mentioned by Read [3]. Self-complementary graphs on a given number of points have been enumerated by Read [4].

Let G be a self-complementary graph and ρ be an inversion of G. The unordered point pairs of any cycle induced by $\rho^{(2)}$ must alternate between being in G and being in $\bar{G}$. Thus each line cycle must have even length. In Section 1 the lengths of line cycles are determined from the lengths of the point cycles. From that discussion it follows that in order that there be no odd line cycles, there can be at most one odd point cycle, and that must have length 1. Moreover a cycle of length $2i$ must be such that i is even. Thus any self-complementary graph must have $4k$ or $4k + 1$ points for some k. If G had $4k$ points, then a point of degree d in G is a point of degree $4k - 1 -$ in $\bar{G}$, which is odd if d is even and vice versa. Thus an Euler self-complementary graph must have $4k + 1$ points for some k.

Given a self-complementary graph G on $4k$ points, we form the graph G^E by adding a new point q and connecting it to each of the points of G of odd degree. First, G^E is Eulerian since G has an even number of points of odd degree. To see that G^E is self-complementary, let σ be an inversion of G and let σ^E be the extension of σ to the points of G^E which leaves q fixed. A point of even degree in G is mapped by σ to a point of odd degree, and vice versa. Thus the lines from q are mapped to nonlines and vice versa by σ^E; so σ^E is an inversion of G^E.

To check that this correspondence is onto, let F be a self-complementary Euler graph and φ an inversion of F. Let v be the unique fixed point of φ, and $F - v$ the graph obtained from F by deleting v and the lines incident to v. Then the restriction of φ to the points of $F - v$ is an inversion of $F - v$, so $F - v$ is self-complementary. Also it is clear that $(F - v)^E \simeq F$, since the points of odd degree in $F - v$ are exactly the ones which were adjacent to v in F. The length of every even point cycle of φ is a multiple of 4, so v is a fixed point of φ^2 while all of the other points are in cycles of even length of

φ^2. Let $\Gamma(F)$ be the automorphism group of F, viewed as a permutation group on the points of F. Then since $\varphi^2 \in \Gamma(G)$, the orbits of $\Gamma(F)$ are unions of the cycles of φ^2. So there is exactly one orbit of $\Gamma(F)$ of odd cardinality, and it contains v. Now suppose $G^E \simeq F$. Let v' be the image of the point q in G^E under an isomorphism of G^E onto F. Then v' is the fixed point of some inversion of F, and so must be in the odd orbit of $\Gamma(F)$ which contains v. Thus $F - v \simeq F - v'$, and of course $F - v' \simeq G^E - q$. But also $G^E - q \simeq G$, as seen just a few lines above. Hence $G \simeq F - v$ whenever $G^E \simeq F$, which shows, finally, that this correspondence is one-to-one.

4. Extensions of Section 1

It is evidently not trivial to pass from $u(x)$ to an enumeration $u(x, y)$ of Euler graphs with lines as well as points for an enumeration parameter. From Fig. 1 it can be seen that the part of $u(x, y)$ corresponding to 6 point graphs is

$$x^6(1 + y^3 + y^4 + y^5 + 3y^6 + 2y^7 + 2y^8 + y^9 + 2y^{10} + y^{11} + y^{12}),$$

which is woefully unsymmetric in the line parameter y. The processes of Section 1 can be extended brutally to accommodate the line parameter, but the result does not promise to be pleasing.

Another direction for generalization is the enumeration of graphs with ny given number of points of even and odd degrees. This can be handled by a quite trivial alteration in the methods of Section 1.

References

1. W. Burnside, *Theory of Groups of Finite Order*, 2nd ed., Theorem VII, p. 191. Cambridge Univ. Press, London and New York, 1911 (reprinted by Dover, New York, 1955).
2. F. Harary, The number of Linear, Directed, Rooted, and Connected Graphs, *Trans. Amer. Math. Soc.* **78** (1955), 445–463.
3. R. C. Read, Euler Graphs on Labelled Nodes, *Canad. J. Math.* **14** (1962), 482–486.
4. R. C. Read, On the Number of Self-Complementary Graphs and Digraphs, *J. London Soc.* **38** (1963), 99–104.