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Ozeki

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On the Problem 1, 2, 3, $\cdots$, $[n^{1/k}] \mid n$

NOBUO OZEKI

Mathematical Institute

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§ 1 Introduction

G. Pólya[1] had proposed a problem of number theory: "If a positive integer is divisible by all the positive integers that do not exceed its square root, it is not greater than 24."

Y. Yanagihara[2] solved this problem. K. Narumi[3] proceeded to the case of cubic root instead of square root, and proved that the integer in question is not greater than 420. And T. Tannaka proved, in the case of 4-th root, that the integer is not greater than 27720.

In the present article, we will determine the values of the largest integers n_5, n_6, $\cdots\cdots$ n_{10}, in the problems of 5-th unto 10-th root, respectively, instead of square root in Pólya's original problem, and give the order of magnitude of n_k in the problem of k-th root, in general.

Before proceeding with our reasoning, we refer here to the outline of Tannaka's method (the methods of Yanagihara and Narumi are based on the same idea). He puts

$$a^4 \leq n < (a+1)^4,$$
$$n = a^4 + \lambda_1 a^3 + \lambda_2 a^2 + \lambda_3 a + \lambda_4, \qquad 0 \leq \lambda_i < a-1.$$

It it is assumed $a \geq 12$, since 27720 ($>12^4$) has certainly the property in question. From $a \mid n$,

$$\lambda_4 = 0,$$

and from $a-1 \mid n$,

$$1 + \lambda_1 + \lambda_2 + \lambda_3 = k_1(a-1), \qquad k_1 < 3.$$

Similar conditions are derived from $a-2 \mid n$, $a-3 \mid n$, and $a-4 \mid n$, respectively. They are solved with respect to the unknowns a, λ_1, λ_2, λ_3, giving first

$$a < 11376.$$

Henceforth various cases are separated and examined, and after a long calculation extending over 3 pages, it is determined that $a=12$ is the unique solution.

This method would require a tremendous labour, if applied to the 5-th root problem.

§ 2 Determination of the largest integers n_k, $k=5 \cdots\cdots, 10$

Lemma 1. (Tchebyscheff's theorem) Let p_k be the k-th prime number, then

$$2p_k > p_{k+1}.$$

First we explain the case of 4-th root. We note that $27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, Let us consider the number

$$n = 27720 \cdot t,$$

and let

$$t=t_1\cdot q_1^{\alpha_1}q_2^{\alpha_2}\cdots\cdots q_k^{\alpha_k},\qquad t_1=2^{\beta_1}\cdots\cdots 11^{\beta_5}$$

be the prime factor decomposition of t, where $q_1=13$, $q_2=17$, and so forth. If $k\geqq 4$, by lemma 1,

$$q_{k+1}^4<2^4\cdot q_k^4<2^7\cdot q_{k-1}^3 q_k<2^9\cdot q_{k-2}^2\, q_{k-1}q_k<2^{10}\cdot q_{k-3}q_{k-2}q_{k-1}q_k$$
$$=1024\cdot q_{k-3}\cdots q_k<27720\cdot t.$$

Therefore

$$q_{k+1}\nmid 27720\cdot t.$$

If $k=3$,

$$q_4^4<2^9\cdot 13\cdot q_1q_2q_3=6656\cdot q_1q_2q_3<27720\cdot t.$$

If $k=2$,

$$q_3^4<2^7\cdot 13^2 q_1q_2=21632q_1q_2<27720\cdot t.$$

If $k=1$,

$$q_2^4=17^4=83521<27720\times 13\cdot t$$
$$27720\times 2=55440>28561=13^4.$$

Hence we know that 27720 is the largest.

Theorem 1. If a positive integer is divisible by all the positive integers that do not exceed its 5-th root, it is not greater than 7 20720. 7 20720 is the largest integer of this natu

Proof. First we note that

$$1,\ 2,\ \cdots,\ 16\mid 7\ 20720=2^4\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13,$$

and that 7 20720 lies between

$$14^5=5\ 37824\qquad\text{and}\qquad 15^5=7\ 59375.$$

We consider the number $n=7\ 20720\cdot t$. Let

$$t=t_1q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k},\qquad t_1=2^{\beta_1}\cdots 13^{\beta_6},$$

where $q_1=17, q_2=19$, and so forth. We can proceed by similar reasoning as above. If $k\geq 5$,

$$q_{k+1}^5<2^{15}\cdot q_{k-4}\cdots q_k=32768\cdot q_{k-4}\cdots q_k<7\ 20720\cdot t.$$
$$\therefore\qquad q_{k+1}\nmid 7\ 20720.t.$$

If $k=4$,

$$q_5^5<2^{14}\cdot 17\cdot q_1\cdots q_4=2\ 78528\cdot q_1\cdots q_4<7\ 20720\cdot t.$$

If $k=3$,

$$q_4^5=29^5=205\ 11149$$
$$7\ 20720\times 17\times 19=2327\ 92560$$
$$\therefore\qquad q_4^5<7\ 20720\cdot t$$

If $k=2$,

$$q_3^{\,5}=23^5=64\ 36343<7\ 20720\cdot t.$$

If $k=1$,

$$q_2^{\,5}=19^5=24\ 76099,$$
$$7\ 20720\times 2=14\ 41440>14\ 19857=17^5.$$

Hence we see that 7 20720 is the largest.
And by similar considerations we obtain following results.

Theorem 2.

k-th root	n_k	
2	$2^3\cdot 3=24$ $4^2 < n_2 < 5^2$ 16 ‖ 25	$1\sim 4 \mid n_2$
3	$2^2\cdot 3\cdot 5\cdot 7=420$ $7^3 < n_3 < 8^3$ 343 ‖ 512	$1\sim 7 \mid n_3$
4	$2^3\cdot 3^2\cdot 5\cdot 7\cdot 11=27720$ $12^4 < n_4 < 13^4$ 20736 ‖ 28561	$1\sim 12 \mid n_4$
5	$2^4\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13=7\ 20720$ $14^5 < n_5 < 15^5$ 537824 ‖ $7\ 59375$	$1\sim 14 \mid n_5$
6	$2^4\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\times 3=122\ 52240\times 3=367\ 56720$ $18^6 < n_6 < 19^6$ $340\ 12224$ ‖ $470\ 45881$	$1\sim 18 \mid n_6$
7	$2^4\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23=53542\ 28880$ $24^7 < n_7 < 25^7$ $45864\ 71424$ ‖ $61035\ 15625$	$1\sim 24 \mid n_7$
8	$2^4\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\times 6=803134\ 33200\times 6=48\ 18805\ 99200$ $28^8 < n_8 < 29^8$ $37\ 78019\ 98336$ ‖ $50\ 02464\ 12961$	$1\sim 28 \mid n_8$
9	$2^4\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31=7220\ 17764\ 46800$ $34^9 < n_9 < 35^9$ $6071\ 69927\ 66464$ ‖ $7881\ 56386\ 71875$	$1\sim 34 \mid n_9$
10	$2^5\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\times 2$ $=5\ 34293\ 14570\ 63200\times 2=10\ 68586\ 29141\ 26400$ $40^{10} < n_{10} < 41^{10}$ $10\ 48576\ 00000\ 00000$ ‖ $13\ 42265\ 93101\ 52401$	$1\sim 40 \mid n_{10}$

Now we explain the method to find out n_k · For instance, the smallest integer div[illegible]e by 1 ……, 5 is $2^2 \cdot 3 \cdot 5 = 60$. It is also divisible by 6. We write this as

$$2^2 \cdot 3 \cdot 5 = 60 \rightarrow 6.$$

By similar procedures, we obtain the following table:

$$2^2 \cdot 3 \cdot 5 \cdot 7 = 420 \rightarrow 7$$
$$2^3 \cdot 3 \cdot 5 \cdot 7 = 840 \rightarrow 8$$
$$2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520 \rightarrow 10$$
$$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 27720 \rightarrow 12$$

………

$$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 122\ 52240 \text{ (8 digits)} \rightarrow 18$$

(×3 must be calculated in this case; see below)

and so forth. On the other hand, we prepare the table of n^k (in Barlow's table, we find these numerals for $n = 1 \sim 100$, $k \rightarrow 10$). These two tables are examined in parallel. First we read the digits of the numerals in the tables. Sometimes, we must multiply the numeral by 3 (as in the case of n_6 or by other small integer.

§ 3 Order of the magnitude of n_k

Next, we will estimate the order of the magnitude of n_k, for general k.

Lemma 2.[5] Let

$$\Psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \Lambda(n), \qquad \Lambda(n) = \begin{cases} \log p & (n = p^m) \\ 0 & (n \neq p^m) \end{cases}$$

then

$$\Psi(x) \geq \frac{x}{4} \log 2.$$

The estimation coefficient (log 2)/4 can be revised. The above inequality is derived from

$$N = \frac{(2n)!}{(n!)^2} = \frac{n+1}{1} \cdot \frac{n+2}{2} \cdots \frac{2n}{n} \geq 2^n.$$

Making use of Stirling formula instead, we obtain

$$\Psi(x) \geq x \log 2 - \frac{1}{2} \log x - \frac{1}{(3x-1)} - \frac{1}{2} \log 2 - \frac{1}{2} \log \pi.$$

Theorem 3. Let 1, 2,……, $[n^{1/k}]$, and let α be the root of $f(x) = 0$, where

$$f(x) = x \log 2 - \frac{1}{2} \log x - \frac{1}{3(x-1)} - \frac{1}{2} \log 2 - \frac{1}{2} \log \pi - 1 - k \log x$$

($\alpha > x_0$ = minimum point of $f(x)$). Then

$$n_k < e\alpha^k.$$

Proof. Suppose that $m_k \rightarrow a$.

$$a^k \leq m_k t < (a+1)^k < ea^k \qquad (a > k),$$
$$\Psi(a) < \Psi(a) + \log t < 1 + k \log a.$$

$f(x)$ has a minimum at x_0, and is monotonically increasing if $x>x_0$. Hence, if it were $a>\alpha$,

$$\Psi(a)>1+k \log a.$$

Therefore we have $m_k<c\alpha^k$.

Numerical calculation for $k=10$ gives $\alpha \doteqdot 64$, $\log \alpha \doteqdot 4.2$, while the corresponding actual value is 3.59.

References

1) Revista Matemática Hispano-Americana, Vol. 2, No. 1, 2.
2) Tokyo Butsuri Gakko Zasshi No. 341.
3) Ibid. No. 355.
4) Ibid. No. 479.
5) Hardy and Wright: An Introduction to the Theory of Numbers, pp. 340～342.