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Research Notes and Comments

The Löschian Numbers As a Problem in Number Theory

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In that part of Lösch's work that can be taken as a contribution to central place theory [*10*, pp. 101-37], the "size" of a good's market area is conventionally measured by the number of complete farms it contains. In practice, since both farms and market areas are hexagons, fractions of farms are encountered, but in every permissible case these fractions sum to a whole number, and hence the size of each market area is always an integer. The complete set of these integers, here denoted by Q, may be termed the Löschian market-area numbers or simply the Löschian numbers. Individual members of Q are here denoted by N.

Elementary geometry shows that the Löschian numbers can be generated systematically by the function

$$Q = x^2 + xy + y^2 ,$$

where x and y are nonnegative integers and $x \leqslant y$ [*4*, *5*, *12*]. An important point overlooked in the literature is that this function does not directly generate values of N. Rather it produces values of D^2 where D denotes the distance between neighboring suppliers of the same good. Fortunately the values of D^2 and N are one and the same; that is, for any particular good

$$D^2 = N.$$

Lösch [*10*, pp. 117-18], without giving proof, stated this relationship in the form

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$$D = d\sqrt{N},$$

where d is the distance, here taken as unity, between adjacent farmsteads. The proof is a straightforward exercise in the geometry of the standard triangular lattice.

A3136

Between $N = 1$ (taken to represent self-sufficiency at each farmstead) and $N = 10{,}000$ inclusive, there occur 2,299 Löschian numbers. An examination of these numbers, and of the non-Löschian numbers that surround them, prompts the conjecture that a given integer is Löschian if, and only if, its prime factorization satisfies the condition that the prime number 2 and all primes of the form $(6k - 1)$, if present, have even powers. Other primes, namely 3 and all primes of the form $(6k + 1)$, may be present with any power; their presence or absence is immaterial to the question of whether or not a given integer is Löschian. This note outlines a proof of this conjecture. The reasoning requires a knowledge of Legendre symbols and Gauss's law of quadratic reciprocity, concepts that are explained in standard number-theory texts [*2*, *3*, *6*, *8*, *9*, *11*].

The Proof

Let d be the greatest common divisor of the integers x and y in $(x^2 + xy + y^2)$. Then $x = ds$ and $y = dt$, where s and t are integers whose greatest common divisor is 1. Thus,

$$\begin{aligned}(x^2 + xy + y^2) &= (ds)^2 + (ds)(dt) + (dt)^2 \\ &= d^2(s^2 + st + t^2).\end{aligned}$$

The square d^2 cannot contain any odd-powered prime factors and may henceforth be ignored. The problem is thus immediately reduced to that of proving that the number $(s^2 + st + t^2)$, in which the greatest common divisor of s and t is 1, is not divisible by an odd power of 2 or of any prime of the form $(6k - 1)$.

Let p^n denote any prime factor of $(s^2 + st + t^2)$. Then p^n divides $(s^2 + st + t^2)$ without remainder, and it is required to show that n must be even for $p = 2$ and $p = (6k - 1)$.

Suppose $p = 2$, $n = 1$, so that $p^n = 2$. Since the greatest common divisor of s and t is 1, s and t cannot both be even numbers. But if either s or t is odd, or if both are odd, then $(s^2 + st + t^2)$ is also odd and thus not divisible by 2. Thus, by reductio ad absurdum, $p^n \neq 2$.

A number of the form $(x^2 + xy + y^2)$, where the greatest common divisor of x and y is greater than 1, may well be divisible by 2^n where n is even. With n even, 2^n is clearly a square and may be written as r^2. Then it is apparent that

$$r^2(s^2 + st + t^2) = (rs)^2 + (rs)(rt) + (rt)^2,$$

which may be written as $(x^2 + xy + y^2)$. In short a number of this general form may be divisible by an even power of 2. An odd power of 2, however, cannot be wholly accommodated in the square r^2; there is always a unit-powered 2 left over, and the argument of the preceding paragraph then applies.

Exactly the same reasoning holds for any prime p for which, given that the greatest common divisor of s and t is 1, the number $(s^2 + st + t^2)$ is not divisible by p^n when $n = 1$. Therefore the remainder of the proof need only show that $(s^2 + st + t^2)$ is not divisible by the first power of any prime of the form $(6k - 1)$.

With p prime such that p^n divides $(s^2 + st + t^2)$, note that $(s^2 + st + t^2) = kp$, where k is some integer. Divide through by t^2 to obtain

$$\left(\frac{s}{t}\right)^2 + \left(\frac{s}{t}\right) + \left(1 - \frac{kp}{t^2}\right) = 0$$

Using the standard formula for the roots of a quadratic equation, this yields

$$\left(\frac{s}{t}\right) = \frac{-1 \pm \sqrt{1 - 4\left(1 - \frac{kp}{t^2}\right)}}{2},$$

whence

$$(2s + t)^2 + 3t^2 = 4kp$$

The left-hand side of this equation may be written as a congruence, modulo p:

$$(2s + t)^2 \equiv -3t^2 \pmod{p}$$

Since $(2s + t)^2$ is a square, $-3t^2$ is a quadratic residue of p. In terms of Legendre symbols,

$$\left(\frac{-3t^2}{p}\right) = +1 = \left(\frac{3}{p}\right)\left(\frac{-1}{p}\right)\left(\frac{t^2}{p}\right) \qquad (1)$$

The next step is to evaluate the Legendre symbols at the right of this equation. There obviously exists some x such that p divides $(x^2 - t^2)$, and hence $(t^2/p) = +1$. Evaluating the term $(-1/p)$ is not quite so simple. With p having the form $(6k - 1)$, two cases must be considered: k odd and k even. The former is presented as an example. To evaluate $(-1/p)$, the standard rule is

$$\left(\frac{-1}{p}\right) = (-1)^B,$$

where $B = \frac{1}{2}(p - 1)$. For $p = (6k - 1)$, $B = \frac{1}{2}(6k - 1 - 1) = (3k - 1)$. With k odd, $(3k - 1)$ is even, thus $(-1/p) = +1$.

With $(t^2/p) = +1$ and $(-1/p) = +1$, it follows from equation (1) that $(3/p) = +1$. Now consider the two primes in the Legendre symbol $(3/p)$ and note that $3 \equiv 3 (\text{mod } 4)$, whereas $p = (6k - 1)$, with k odd, implies that $p \equiv 1 (\text{mod } 4)$. Therefore, by the law of quadratic reciprocity,

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = +1. \tag{2}$$

Now, still supposing $p = (6k - 1)$, consider the congruence $x^2 \equiv p (\text{mod } 3)$. Substitute $p = (6k - 1)$ to obtain $x^2 \equiv (6k - 1)(\text{mod } 3)$. Since 3 divides $6k$, this in turn may be written as $x^2 \equiv -1 (\text{mod } 3)$, which is equivalent to $x^2 \equiv 2 (\text{mod } 3)$. By inspection, there is no integer x that satisfies this congruence. In other words, no prime of the form $(6k - 1)$ is a quadratic residue of 3. Thus $(p/3) = -1$. However, this contradicts the law of quadratic reciprocity as expressed in equation (2). Therefore, by reductio ad absurdum, $p \neq (6k - 1)$ with k odd.

Application of the same line of reasoning to cases where k is even leads, by a slightly different route, to the parallel conclusion that $p \neq (6k - 1)$ with k even. This completes the proof.

The procedures used to prove the inadmissibility of primes of the form $(6k - 1)$ may equally be used to demonstrate the acceptability of primes of the form $(6k + 1)$. The remaining case of $p = 3$ is trivial, since it is obvious without any calculation that -3 behaves as a quadratic residue of 3. The key to the whole problem lies in the fact that -3 is a quadratic residue of primes of the form $(6k + 1)$ and a quadratic non-residue of primes of the form $(6k - 1)$. This fact is proved in modern texts [*8*, p. 75; *3*, pp. 74–75] and was indeed known to Gauss [*7*, p. 77]. However, the relationship between this result and the problem of characterizing the Löschian numbers is not instantaneously obvious.

The argument above proves that the prime factorization of any Löschian number N does not contain an odd power of 2 or of any prime of the form $(6k - 1)$. Using similar methods, it is also possible to prove the converse: an integer having a prime factorization such that 2 and primes of the form $(6k - 1)$, if present, have even powers, whereas 3 and primes of the form $(6k + 1)$ have any power, is of the form $(x^2 + xy + y^2)$. A strong conclusion is thus reached. Not only does every Löschian number have a certain type of prime factorization, but also every integer with that type of prime factorization is a Löschian number.

Implications

Given the above findings, several corollaries follow immediately.

Corollary 1. There are infinitely many Löschian numbers, and hence an infinite number of theoretically possible market-area sizes.

Corollary 2. Given that odd and even integers are equally numerous, the probability that a Löschian number is odd is 0.75, and the probability that it is even is 0.25. This follows from the fact that $(x^2 + xy + y^2)$ is even only if x and y are both even.

Corollary 3. The greatest common divisor and the least common multiple of any two or more Löschian numbers are also Löschian numbers.

Corollary 4. The product of two Löschian numbers is always a Löschian number.

Corollary 5. The product of a Löschian number and a non-Löschian number is never a Löschian number.

Corollary 6. The product of two non-Löschian numbers is sometimes Löschian and sometimes not.

The general theorem itself may be used to answer the question of whether or not any given integer is Löschian. The integer must first be reduced to its prime factors, a step that is facilitated by the use of published factor tables [*1*, pp. 844–63]. Then, if 2 or any prime of the form $(6k - 1)$ is present with an odd power, the number is not Löschian; otherwise, it is Löschian. Is 2,691 a Löschian number? No: its prime factorization is $3^2 \cdot 13 \cdot 23$, and 23 has the form $(6k - 1)$ with an odd power. Is 2,692 Löschian? Yes: its factors are $2^2 \cdot 673$. In this way the admissibility of any number may rapidly be established.

Transferred from the realm of number theory to Lösch's isotropic plain, the terms x and y in $(x^2 + xy + y^2)$ have concrete expression as grid coordinates giving the location of a supplier relative to the origin at the central metropolis [*12*, pp. 114–16]. Given that N is a Löschian number, it would therefore be of interest to know the corresponding values of x and y. These values can of course be found iteratively, but a quicker method is desirable. Given the value of N, the need is for a general formula to extract solutions to the Diophantine equation $N = (x^2 + xy + y^2)$. An attack on this problem has so far been unsuccessful, but it may be premature to conjecture that the problem is insoluble.

LITERATURE CITED

1. Abramowitz, M., and I. A. Stegun, eds. *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards, United States Department of Commerce, 1964.
2. Andrews, G. E. *Number Theory.* Philadelphia: W. B. Saunders, 1971.
3. Bolker, E. D. *Elementary Number Theory: An Algebraic Approach.* New York: W. A. Benjamin, 1970.
4. Dacey, M. F. "A Note on Some Number Properties of a Hexagonal Hierarchical Plane Lattice." *Journal of Regional Science,* 5, No. 2 (1964), 63–67.
5. ———. "An Interesting Number Property in Central Place Theory." *Professional Geographer,* 17, No. 5 (1965), 32–33.
6. Davenport, H. *The Higher Arithmetic.* 4th ed. London: Hutchinson, 1970.

7. Gauss, C. F. *Disquisitiones Arithmeticae.* Trans. A. A. Clarke, S. J. New Haven: Yale University Press, 1966.
8. Hardy, G. H., and E. M. Wright. *An Introduction to the Theory of Numbers.* 4th ed. London: Oxford University Press, 1960.
9. Jones, B. W. *The Theory of Numbers.* New York: Holt, Rinehart and Winston, 1955.
10. Lösch, A. *The Economics of Location.* Trans. W. H. Woglom and W. F. Stolper. New Haven: Yale University Press, 1954.
11. McCoy, N. H. *The Theory of Numbers.* New York: Macmillan, 1965.
12. Tarrant, J. R. "Comments on the Lösch Central Place System." *Geographical Analysis,* 5 (April 1973), 113–21.