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correspondence
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October 10, 1994

Dr. Neil J. A. Sloane
AT&T Bell Laboratories
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600 Mountain Avenue
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Dear Neil,

I probably forgot to tell you about the sequence $1, 1, 2, 4, 14, 62, \ldots$ that I published in the MONTHLY long ago (April 1974, page 340). It counts "necklace permutations," a fairly natural kind of combinatorial object that I hope somebody will soon enumerate.

Cordially,

Don

Donald E. Knuth
Professor

DEK/pw

→ to Neil J A Sloane

3322

February 26, 1973

Prof. David A. Klarner
Computer Science Dept.
Stanford University

Dear David:

Last week while skiing in Norway I thought of another enumeration problem I couldn't solve, but it looks interesting so I wonder if you will see how to do it.

The problem is to discover the number of different "necklace permutations" -- this is a word I made up, you can change it if you wish. It represents the number of essentially different orders in which a person can change n white beads of a necklace into all black beads, not counting the operation of turning and/or flipping over the necklace whenever such an operation preserves the current black/white pattern.

Thus, $a_1 \ldots a_n$ is equivalent to $(\epsilon a_1+j) \ldots (\epsilon a_k+j) a_{k+1} \ldots a_n$, modulo n, whenever both are permutations, for all $\epsilon = \pm 1$, and $1 \le j, k \le n$; and transitivity applies too. In particular, $a_1 \ldots a_n$ is always equivalent to $a_2 a_1 a_3 \ldots a_n$ and to $a_1 \ldots a_{n-2} a_n a_{n-1}$. The distinct necklace permutations for $n \le 6$ are

1;

12;

123;

1234, 1324;

12345, 12435, 13245, 13425;

123456, 123546, 124356, 124536, 124635, 132456, 132546,
134256, 134526, 134625, 135246, 142356, 142536, 142635;

and if my quick count isn't wrong there are 62 of order 7.

Sincerely,

Donald E. Knuth
Professor

DEK/pw

1, 1, 1, 2, 4, 14, 62,

~~[illegible]~~ [from Wilf ~~[illegible]~~]
Nov 7, 90

Dear Dan:

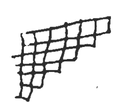
Thanks for unearthing the letters to Kleitman and Klarner, from 1972-3. I was aware of the fact that the # of perms (!) that contain a fixed pattern is "nearly", but not quite, independent of the pattern. I made tables about 5 yrs ago, and conjectured that if the pattern of length k is fixed, and $f(n)$ is the # of n-perms that contain the pattern, then the asymptotic expansion of $f(n)$, for $n \to \infty$, will have a first term that is independent of the pattern. This is still unsettled. I've thought about it & asked others about it. The asymptotics is known if the pattern = identity, but even that is hard. It's all tied to stack-sortable permutations.

Your other letter, to Klarner, has just disrupted my lunch hour. It's a very pretty question, to enumerate "necklace permutations."

First let me transform the question.

Fix n. Let Π_n be the set of all necklaces of n beads of 2 colors (equivalence classes under, say, dihedral group, as you proposed). Make Π_n into a partially ordered set, as follows: a necklace ν' is covered by a necklace ν'' if ν' can be transformed into ν'' by a single bead-blackening. The figure on p. 1, enclosed, shows Π_6.

Then your question is: how many paths are there, in Π_6, from  to  ? (when $n=6$).

This kind of question has led to many deep combinatorial results. For the "weak Bruhat order" of n-permutations, where ~~[illegible]~~ "$\sigma' \leq \sigma$" means that σ'' can be obtained from σ' by a finite sequence of adjacent transpositions, each of which increases the # of inversions, let $f(n)$ = # of paths from $(12 \cdots n) \to (n\ n-1 \cdots 1)$ 

Then Stanley "observed" that $f(n)$ = # of Y.T. on the shake & proved it algebraically. Edelman & Greene proved it bijectively. It then turned out that the whole thing was implicitly contained in some older work of Schütz. on plactic monoids.

To get back to this question, I checked Sloane to see if 1,1,1,2,4,14,62,... is the beginning of some famous sequence, & it isn't.

Next, on page 1 enclosed I sketched π_6 and its paths from bottom to top, as you see. Note also that if H is the covering matrix of the partial order, then (H^n) is equal to the number of paths in question, as illustrated on page 2, enclosed.

Naturally I haven't <u>answered</u> <u>anything</u>. But somehow I feel like I know what the question "really" is. It suggests that instead of diving in first to the deeper waters of the dihedral group, why not kill the flip, and stick with cyclic equivalence. That one is plenty hard, still, but might be do-able. The sequence begins 1,1,1,2,4,23,... (cf. Sloane)

Curtis Greene has done not only the number of maximal chains from bottom to top in the wk. Bruhat order, but also in the "shuffle" poset. So maybe I'll ask him if he knows any more general theorems.

Anyway, that's what happened during my lunch break today.

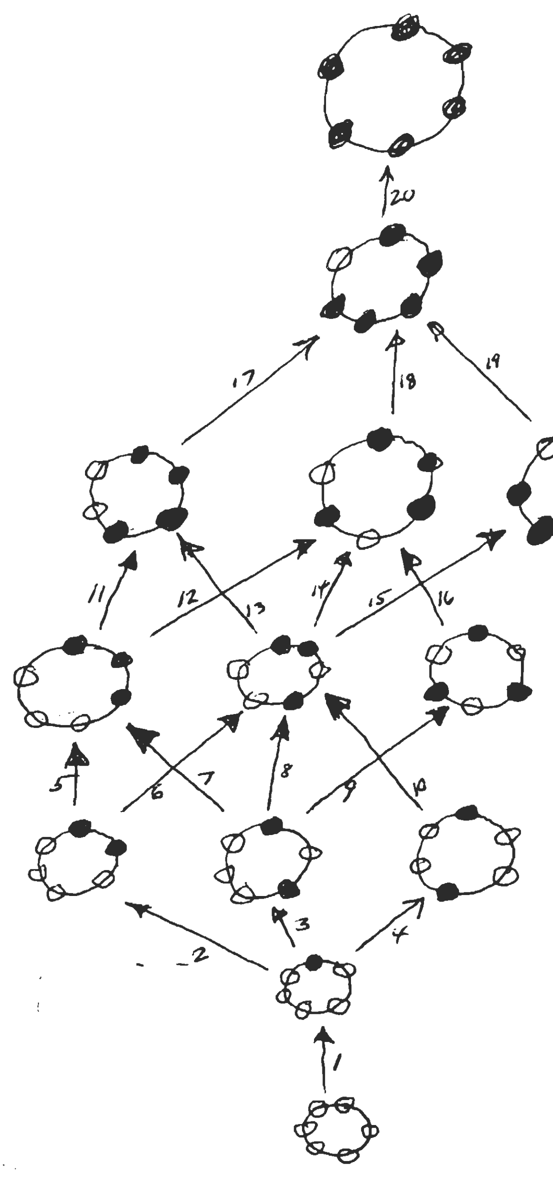
Best,

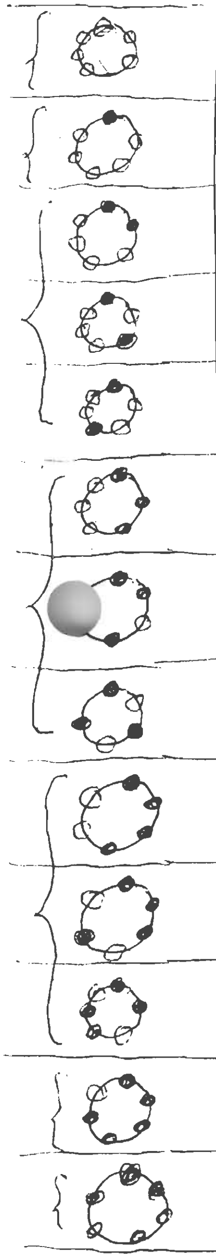
Herb. (Wilf)

1, 2, 5, 11, 17, 20
——— 12, 18, 20
—, 6, 13, 17, 20
———, 14, 18, 20
———, 15, 19, 20
1, 3, 7, 11, 17, 20
———, 12, 18, 20
—, 8, 13, 17, 20
———, 14, 18, 20
———, 15, 19, 20
—, 9, 16, 18, 20
1, 4, 10, 13, 17, 20
———, 14, 18, 20
———, 15, 19, 20

the 14 paths from bottom to top (path = sequence of edge numbers)

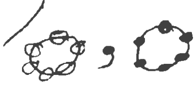
n = 6





	1	2	3	4	5	6	7	8	9	10	11	12	13
1		1											
2			1	1	1								
3						1	1						
4						1		1					
5							1						
6									1	1			
7									1	1	1		
8										1			
9												1	
10												1	
11												1	
12													1
13													

A matrix, for $n = 6$

(A^6)  $= 14$

November 8, 1990

Dan!

Here is another installment of ruminations on the partially ordered set of necklaces of n beads (read this *after* the batch of handwritten stuff I sent you yesterday).

When I looked at that poset, it occurred to me was that I didn't even know the numbers of necklaces in its layers, i.e., the numbers of necklaces of n beads, each black or white, with exactly k black beads. This is quite a fundamental number, because it's the number of subsets of k unlabelled things chosen from a set of n unlabelled things on a circle. So it is a *circular binomial coefficient*, which is really quite nice.

I'll denote it by $\binom{n}{k}_{\mathcal{C}_n}$, where the subscript indicates that the operative group is the cyclic group. The layer counts in your original question would be $\binom{n}{k}_{\mathcal{D}_n}$, for the dihedral group. The original binomial coefficients belong to the identity group $\mathcal{E}_n$, etc. etc. (I think it's delightful that all of a sudden there are mountains of new binomial coefficients to play with).

In general, if $\mathcal{G}_n$ is any subgroup of S_n then by Pólya's theorem one has the generating function (I'm following Harary *Graphical Enumeration*, p. 36)

$$\sum_k \binom{n}{k}_{\mathcal{G}_n} x^k = Z(G, 1+x) \tag{1}$$

where Z is the cycle index of G. Precisely, $Z = Z(s_1, \ldots, s_n)$ is a polynomial in n variables, and we are to substitute $1 + x^j$ for each s_j on the right side of (1).

In particular, since

$$Z(C_n, s) = \frac{1}{n} \sum_{d \backslash n} \phi(d) s_d^{n/d}$$

one has

$$\sum_k \binom{n}{k}_{C_n} x^k = \frac{1}{n} \sum_{d \backslash n} \phi(d)(1 + x^d)^{n/d}$$

and therefore the evaluation

$$\binom{n}{k}_{C_n} = \frac{1}{n} \sum_{d \backslash (n,k)} \phi(d) \binom{n/d}{k/d}.$$

The beginnings of the cyclic Pascal triangle are as follows:

1
1 1
1 1 1
1 1 1 1
1 1 2 1 1
1 1 2 2 1 1
1 1 3 4 3 1 1
1 1 3 5 5 3 1 1
1 1 4 7 10 7 4 1 1
1 1 4 10 14 14 10 4 1 1
1 1 5 12 22 26 22 12 5 1 1

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Dennis White has a theorem, that I'll look up, to the effect that Pólya theory tends to make unimodal sequences. It might imply already that the rows are always unimodal.

This is really pretty stuff, and I'm glad you sent me your original question.

Herb (Wilf)

fa1,

3322

Neil

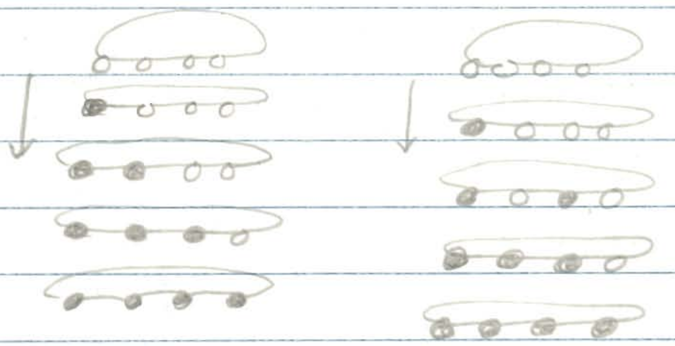
1 2 3 4 5 6 7

1 2 4 14 62 . . . .

Necklace permutations   Am Math Monthly 81 p340 (1974)

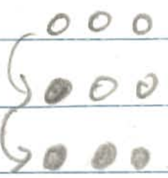
(i.e. # of distinct ways of successively coloring all the beads of a necklace, ignoring rotation & flipping.

eg $n = 4$



$\# = 2$.

$n = 3$



$\# 1$

Colin

(Mallows)