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ENUMERATING UNLABELLED HAMILTONIAN CIRCUITS

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When considering Hamiltonian circuits (HC's) in a graph G which has a high degree of symmetry, one may want to consider HC's as being equivalent under the automorphisms of G and under the cyclic shifts and reversals of a circuit. We refer to these equivalence classes as unlabelled HC's. In the first part of this paper, I discuss these concepts and their elementary properties. Then I discuss methods of enumerating unlabelled HC's, the results I have obtained for the Platonic solids and for the n-dimensional octahedron, and some other known results.

I. DEFINITIONS AND BASIC PROPERTIES.

Let G be a simple graph, i. e. without loops or multiple edges. We label the points of G by the integers 1, 2, ..., p = $|G|$ in some manner.

1. DEFINITION. A labelled Hamiltonian circuit of G is a permutation $(v_1, v_2, \ldots, v_p)$ of the vertices of G such that $v_i v_{i+1}$ is an edge of G for each i, where i + 1 is taken (mod p) (i. e. $v_p v_1$ is also an edge of G).

Let L denote the set of labelled HC's of G. Recall that a permutation f of the symmetric group S_p is an automorphism of G means that ij is an edge of G if and only if f(i)f(j) is an edge of G. The automorphisms of G form a group A. In our examples, A will be the symmetry group of some polytope. An automorphism f of G acts on L as follows:

$$f(v_1, v_2, \ldots, v_p) = (f(v_1), f(v_2), \ldots, f(v_p)).$$

We shall generally consider A as a group acting on L. Thus A induces an equivalence relation on L and we denote the set of equivalence classes by L/A.

Two HC's may also be equivalent under cyclic shifts and reversals of circuits. Let s, r be the functions on L defined by:

$$s(v_1, v_2, \ldots, v_p) = (v_2, v_3, \ldots, v_p, v_1);$$

$$r(v_1, v_2, \ldots, v_p) = (v_p, \ldots, v_2, v_1).$$

Let D be the group generated by s and r. D has 2p elements and is the dihedral group D_p of symmetries of a regular p-gon.

D and A generate a group $B = A \times D$ of bijections on L. Further, the elements of D and of A commute so that B is a subgroup of $A \otimes D$ and both A and D are normal subgroups of B. B induces an equivalence relation on L which includes the relation induced by A. We denote the set of equivalence classes by L/B.

2. DEFINITION. _An unlabelled Hamiltonian circuit of G is an equivalence class of L/B._

We let U = L/B be the set of unlabelled HC's of G.

Before continuing, let us observe that B can be a proper subgroup of $A \otimes D$. Consider $G = K_3$ (= C_3), the complete graph on 3 vertices (= the 3-cycle). Then L consists of all permutations of 3 points and B = A = D = S_3 is the symmetric group on 3 elements. For example, s(1,2,3) = (2,3,1), so that (1,2,3) ∿ (2,3,1) by an action of D, but the permutation f = (1,2,3) has f(1,2,3) = (2,3,1) so that these circuits are already equivalent by an action of A. Here we have $|L| = 6$, $|L/A| = |U| = 1$.

3. PROPOSITION. U = L/B = (L/A)/(B/A), where the right hand side is the set of equivalence classes of L/A with respect to the equivalence relation induced on L/A by the quotient group B/A.

The proof is straightforward and left to the reader.

3.1. COROLLARY. $|L/A|/2p \leq |L/B| \leq |L/A|$.

Proof. This follows since $1 \leq |B/A| \leq |D| = 2p$.

4. PROPOSITION. All the classes in L/A have $|A|$ elements.

Proof. Let ℓ be a HC in L and let $A\ell = \{f(\ell) | f \in A\}$ be the equivalence class of ℓ in L/A. The mapping from A to Aℓ given by $f \to f(\ell)$ is surjective. It is also injective, since if $f(\ell) = g(\ell)$, then $f(v_i) = g(v_i)$ for each vertex v_i, hence f = g.

4.1. COROLLARY. $|L| = |A| \cdot |L/A|$ and $|L|/2p|A| \leq |U| \leq |L|/|A|$.

Since both A and D are normal in B, it is possible to interchange the roles of A and D in the above results and in the next Section. This may be more natural when $|A|$ is small, such as in the problem of knight's circuits on an $n \times n$ chessboard, where $|A| = 8$ and $|D| = 2n^2$. In this context, authors have always referred to the equivalence classes L/D as being the HC's.

The ideas of this Section have been implicit in previous work on the n-cube (2) and on the Platonic solids (1, pp. 262-266; 4; 6).

II. COMPUTATIONAL METHODS.

The problem of counting HC's on a graph G breaks into several types of problem depending upon whether one wants labelled HC's or unlabelled ones or both and upon whether one wants the actual HC's or just the number of them. In any case, it is generally sufficient to determine L/A rather than L and this reduces our work by a factor of $|A|$. In this section, we discuss methods of computing L/A and of obtaining U from it.

The first method consists of considering classes of L/A as being represented by an element of L. We generate elements of L by some method, usually a "back-track" method, and we restrict the output to elements which are distinct under the action of A. The restict-ion can be one or a combination of three types. These types can be used in any kind of backtrack process (12; 5). I will describe them all here although I have only used the first in computing L/A; the other types have been used in computing U.

A. Initial restriction. Here we require the HC to begin in a certain way. To illustrate, consider the cube in Figure 1. The cube has $|A| = 48$. We can move

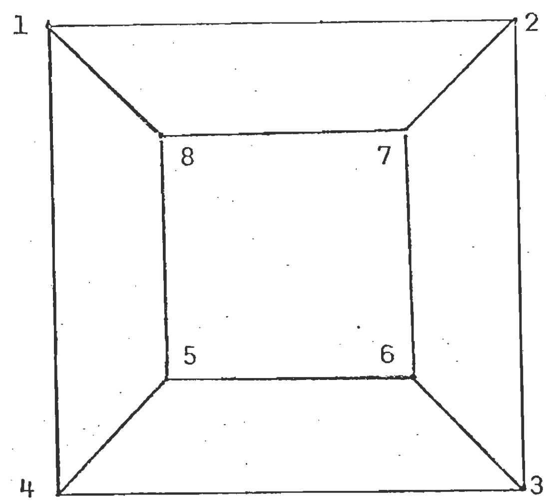

FIGURE 1.

the cube so that our first vertex $v_1 = 1$. Then we can rotate the cube about the axis 16 so that $v_2 = 2$ and we can reflect about the plane 1256 so that $v_3 = 3$. We can then complete the HC in only two ways, so $|L/A| = 2$ and $|L| = 96$. This kind of restriction can be easily incorporated into any standard algorithm for finding HC's such as that of [13]. This restriction was sufficient for studying the Platonic solids, yielding the results stated in [9] and in Section III.

B. Rejection during generation. If an initial segment, which has been generated, is equivalent under an action of A to an initial segment which has previously been considered, we reject it and hence any circuit having this initial segment. Such testing will slow down each step of the backtracking, but will reduce the number of possibilities considered. One must balance these factors in deciding how much testing to carry out at each stage. Incorporating such tests into any standard algorithm is not difficult. Such rejections were used in dealing with the n-octahedron, see Section IV.

C. Rejection after generation. If a HC is equivalent by an action of A to one previously found, we reject it. Such testing is essentially added on to the end of the generation algorithm. The testing must be sufficient to detect any equivalences not covered by types A and B.

One can carry out this testing in two ways. The first method assumes that we have stored a list of all circuits generated. Then we work through the list, taking the first remaining element and applying the actions of A (or those not covered by previous restrictions) to it. The resulting equivalent circuits are deleted from the list. When we finish, we have a list of distinct representatives for L/A. I call this a sieving method, by virtue of its similarity to the sieve of Erastothenes. I used this technique for the Platonic solids. This technique can only be applied when the list of generated circuits is small enough to store.

The second method of rejection after generation requires that the HC's are being generated in some known order, usually the lexicographic order of the backtrack algorithm. As each new circuit is generated, we apply the actions of A (or those not previously covered) to it. If any action gives a circuit which precedes the present one in the order of generation, then the present one is not new and is rejected. I used this method for the n-octahedron.

We can also compute L/A by an entirely different approach. In this method we represent a HC in some other way than as a permutation of vertices. We do this in such a way that HC's which are equivalent under A have a small number of representations, hopefully only one. In fact, Hamilton's first work (6; 4; 1, pp. 262-266) on HC's on the dodecahedron used this approach. As one proceeds along a HC and arrives at a vertex, there are two edges leading out, one to the right and one to the left. If we denote these as r and ℓ, then a HC is represented by a 20-tuple of r's and ℓ's, subject to certain conditions arising from the relations $\ell^5 = r^5 = 1$, $r\ell r = \ell r r \ell$, etc. Such a sequence is invariant under the proper symmetries of the dodecahedron, while the improper symmetries simply interchange r and ℓ. Hamilton's work is notable as the first presentation of a group by generators and relations. A similar method can be applied to the other Platonic solids (4).

A related representation has been used for the n-cube (3; 8). On the n-cube, the n edges at a vertex correspond to the n dimensions, so we can denote the edge along which the HC leaves a vertex by an integer from 1 to n. Then a HC is represented by a 2^n-tuple of integers from 1 to n, subject to certain conditions. Unfortunately there are still n! such sequences for each class of L/A.

In Section IV, I describe a representation for the n-octahedron which gives only one representative for each class in L/A.

In any case, we must generate the representatives by some algorithm. If there are several representatives for a class of L/A, then we must apply some combination of our restrictions A, B, C to obtain just one representative for each class.

Now consider the problem of finding U = L/B. If we are not finding L/A, then the above discussions can be carried over to L/B simply by replacing A by B. If we are finding L/A, then we can find L/B by rejection after generation, as described in C above, but we need only test for equivalence under the actions of B/A = $(A \times D)/A = D/(A \cap D)$, or more broadly, under the actions of D.

III. PLATONIC SOLIDS.

I have applied the first method of computing L/A and the sieving method of rejection after generation to find L/B = U for the regular polyhedra in three dimensions. The results are given in Table 1.

TABLE 1.

G	|A|	|L/A|	|L|	|U|
Tetrahedron	24	1	24	1
Cube	48	2	96	1
Dodecahedron	120	10	1200	1
Octahedron	48	4	192	2
Icosahedron	120	256	30720	17

The values of $|U|$ given in Table 1 agree with those given in [4].

IV. THE N-DIMENSIONAL OCTAHEDRON.

The graph of the n-dimensional octahedron (or cross polytope) is the complete n-partite graph $K_{2,2,\ldots,2}$. That is, it consists of n pairs of points and every point is connected to every other point except the other point of its pair. The paired points will be called antipodal. In [10], I have shown that

$$|L| = \Sigma_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2n}{2n-k} 2^k (2n-k)!$$

and that $|L| \sim (2n)!/e$.

The symmetry group of the n-octahedron has $|A| = 2^n n!$ elements corresponding to the n! permutations of pairs and the 2^n interchanges within pairs. From the point of view of one point, all other points are equivalent except its antipode. The actions of A upon a HC will arbitrarily permute the vertices, except that antipodal points are always carried to antipodal points. From this, we see that an element of L/A is completely determined by knowing which positions of a HC contain antipodal points. To illustrate, consider the 3-octahedron in Figure 2, and consider the HC, $\ell =$ (1,2,3,4,5,6). The antipodal points are in the pairs of positions (1,4), (2,5), (3,6) and these three pairs characterize $A\ell$, the equivalence class of ℓ in L/A.

Using this observation, we define the <u>difference vector</u> $(d_1,d_2,\ldots,d_{2n})$ of a HC $(v_1,v_2,\ldots,v_{2n})$ by letting d_i be the distance from v_i to its antipodal point

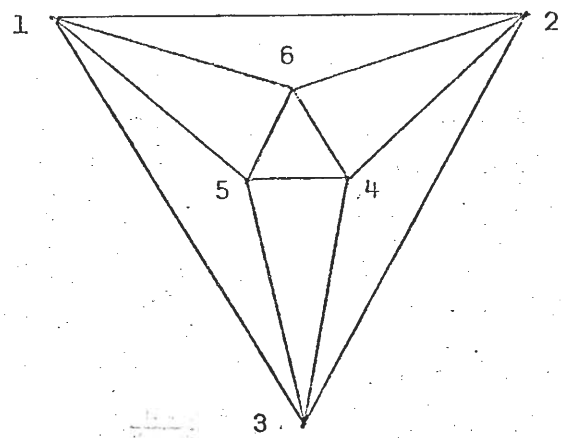

FIGURE 2.

v_{i+d_i}, measured along the HC. That is, if v_i and v_j, $i < j$, are antipodes, then $d_i = j-i$ and $d_j = 2n+i-j$. Two HC's are equivalent under A if and only if they have the same difference vector. For example, the difference vector of the HC ℓ of Figure 2 is (3,3,3,3,3,3). The other difference vectors for the 3-octahedron are: (2,3,4,2,3,4), (3,4,2,3,4,2), (4,2,3,4,2,3).

We have that a vector of integers $(d_1,d_2,\ldots,d_{2n})$ is a difference vector if and only if

a) $2 \leqq d_i \leqq 2n-2$;

b) $d_i + d_{i+d_i} = 2n$.

(Here $i + d_i$ is taken (mod 2n).) Further, the generating actions s and r of D have simple effects of difference vectors:

$$s(d_1, \ldots, d_{2n}) = (d_2, \ldots, d_{2n}, d_1);$$

$$r(d_1, \ldots, d_{2n}) = (2n-d_{2n}, \ldots, 2n-d_1).$$

Since $|L|$ is known, we can restrict the generation of difference vectors. After some experimentation, the following conditions were incorporated during generation:

a) $d_1 = \min d_i$.

b) If $d_i = d_1$, then $d_{i+1} \geqq d_2$.

c) If $d_1 = 2$, then $d_{i+1} \leqq n$.

d) If $d_i = d_1 = 2$, then $d_{i+1} \leqq 2n-d_2$.

For n = 8, these restrictions reduce the number of difference vectors generated by a factor of 19.6, thus to only about 1.56 $|U|$. The results for $2 \leqq n \leqq 8$ are given in Table 2 where $|VG|$ is the number of difference vectors generated and R is the ratio $4n|U||A|/|L|$. The program was written in Algol and took about 8 minutes on an ICL 1905E. The values of R lead me to:

5. CONJECTURE. $|U| \sim |L|/4n|A|$.

(Compare with Corollary 4.1 which gives $|U| \geqq |L|/4n|A|$.)

V. SOME OTHER KNOWN RESULTS.

For the n-cube, $|U| = 1$ for n = 2, 3 and $|U| = 9$ for n = 4 (8). I have the impression that the value for n = 5 is known but I cannot locate it. Not so. |L| is known.

A number of results for the problem of finding knight's circuits on a chess board are given in (1, pp. 174-185; 7). For a square n × n board, n must be even for any circuits to exist and there are none for n = 4. Duby (2) and Stone (11) independently found $|L/D| = 9862$ for the case n = 6, using a method similar to that given

TABLE 2.

n	|L|	|A|	|L/A|	|U|	|VG|	R
2	8	8	1	1	1	8
3	192	48	4	2	2	6
4	11,904	384	31	7	7	3.614
5	1,125,120	3,840	293	29	31	1.980
6	153,262,080	46,080	3,326	196	255	1.414
7	28,507,207,680	645,120	44,189	1,788	2,666	1.133
8	6,951,513,784,320	10,321,920	673,471	21,994	34,392	1.045

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in {13}. For the case n = 8, Duby found 75,000 circuits having the same first 35 moves and he cautiously expressed "a strong belief that the total number ... can be over one million". Stone and I would estimate about $10^{23\pm3}$ such circuits on a heuristic probabilistic basis.

In such a situation of an essentially uncomputable number of HC's, one might maturally ask only for circuits having some interesting property, such as being invariant under A or part of A. For a square board, $|A| = 8$, but there are no circuits invariant under all of A. On the 6 × 6 board, there are ten elements of L/D which are invariant under rotations of the board. Five of these are shown in {7}, the other five being their reflections. I do not know if such circuits have been enumerated on the 8 × 8 board.

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