Absolute Primes

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For a long time primes have attracted the attention of mathematicians, especially those primes that possess some sort of symmetry. The mysterious and inconceivable repunits

$$A_n = 111 \ldots 1_{(n)},$$

whose decimal representation contains only units, form an important class of them[1]. For a repunit A_n to be prime, the number n of digits in its decimal representation must be also prime. But this condition is far from being sufficient: for instance, $A_3 = 111 = 3 \cdot 37$ and $A_5 = 11111 = 41 \cdot 271$. Some repunits are nonetheless prime: A_2, A_{19}, A_{23}, A_{317} and possibly A_{1031}, give the examples. The question of primeness of repunits was discussed by M. Gardner [1] and later in [2-4]. It is completely unclear whether the number of prime repunits is finite or infinite.

The prime repunits are examples of integers which are prime and remain prime after an arbitrary permutation of their decimal digits. Integers with this property are called either *permutable primes* according to H.-E. Richert [5], who introduced them some 40 years ago, or *absolute primes* according to T.N. Bhagava and P.H. Doyle [6], and A.W.Johnson [7]. The intent of this note is to give a short proof, which does not require significant number crunching, of all known facts referring to absolute primes different from repunits.

Analyzing the table of primes which are less than 10^3, we find 21 absolute primes different form the repunit 11:

$2, 3, 5, 7, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991.$

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[1]By a subscript in brackets we will indicate the number of digits in the decimal representation of the integer.

The first observation is easy to get:

Lemma 1: A multidigit absolute prime contains in its decimal representations only the four digits 1, 3, 7, 9.

Proof: If any of the digits 0, 2, 4, 5, 6, 8 appear in the representation of an integer, then by shifting this digit to the units place we get a multiple of 2 or a multiple of 5.

Now we can confine the area of the search, and this helps us to proceed with the following deliberations.

Lemma 2: An absolute prime does not contain in its decimal representation all of the digits 1, 3, 7, 9 simultaneously.

Proof: Let N be a number with all of the digits 1, 3, 7, 9 in its decimal representation. Let us shift these four digits to the rightmost four places, to obtain an integer

$$N_0 = \overline{c_1 \ldots c_{n-4} 7931} = L \cdot 10^5 + 7931,$$

where the notation $\overline{a_1 \ldots a_n}$ is used to denote the number $a_1 10^{n-1} + a_2 10^{n-2} + \ldots + a_{n-1} 10 + a_n$, which decimal representation consists of digits $a_1, \ldots, a_n$. The integers $K_0 = 7931$, $K_1 = 1793$, $K_2 = 9137$, $K_3 = 7913$, $K_4 = 7193$, $K_5 = 1937$, $K_6 = 7139$ have different remainders on dividing by 7 since $K_i \equiv i \pmod 7$. The seven integers $N_i = L \cdot 10^5 + K_i$ for $i = 0, 1, 2, 3, 4, 5, 6$ also have different remainders on dividing by 7. Therefore one of them is a multiple of 7. Since these integers can be obtained from N by a permutation of digits, N is not an absolute prime.

Lemma 3: No absolute prime contains in its decimal representation three digits a and two digits b simultaneously, provided $a \neq b$.

Proof: Suppose that an integer N contains digits a, a, a, b, b in its decimal representation. By a permutation of digits of N, we can obtain integers

$$N_{i,j} = \overline{c_1 \ldots c_{n-5} aaaaa} + (b - a)(10^i + 10^j),$$

where $4 \geq i > j \geq 0$. Since the integers $10^4 + 10^1$, $10^3 + 10^2$, $10^3 + 10^1$, $10^2 + 10^0$, $10^1 + 10^0$, $10^4 + 10^0$, $10^4 + 10^2$ yield different remainders on dividing by 7, which are respectively $0, 1, 2, 3, 4, 5, 6$, so do the integers $(b - a)(10^i + 10^j)$, when $4 \geq i > j \geq 0$. Therefore among the integers $N_{i,j}$ there exists one that is a multiple of 7.

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Using these two lemmas we are able by direct calculation (preferably on a computer) to prove that no n-digit absolute primes exist with $n = 4, 5, 6$. For example, if $n = 6$, we have to check all the numbers

$$\overline{aaaaab}, \quad \overline{aaaabc}, \quad \overline{aabbcc},$$

where $a, b, c \in \{1, 3, 7, 9\}$.

Lemma 4: If $N = \overline{c_1 \ldots c_{n-6}aaaaab}$ is an absolute prime, then $K = \overline{c_1 \ldots c_{n-6}}$ is divisible by 7.

Proof: By permutations of the last 6 digits of N we can obtain the integers

$$N_i = K \cdot 10^6 + a \cdot A_6 + (b - a)10^i$$

for $0 \le i \le 5$. Since $b - a$ is even and the powers 10^i, $0 \le i \le 5$, have different nonzero remainders on dividing by 7:

$$10^0 \equiv 1, \quad 10^1 \equiv 3, \quad 10^2 \equiv 2, \quad 10^3 \equiv 6, \quad 10^4 \equiv 4, \quad 10^5 \equiv 5,$$

the integers $(b - a)10^i$ have the same property. If the integer $K \cdot 10^6 + a \cdot A_6$ had a nonzero remainder on dividing by 7, we would find an integer $(b-a)10^i$, which has the opposite remainder, and get N_i which is divisible by 7. Since this is impossible, the number $K \cdot 10^6 + a \cdot A_6$ is a multiple of 7. Moreover, as $A_6 = 10^0 + 10^1 + 10^2 + 10^3 + 10^4 + 10^5 \equiv 1 + 3 + 2 + 6 + 4 + 5 = 21 \equiv 0 \pmod 7$, we conclude that $K \cdot 10^6$ and hence K, is divisible by 7.

Theorem 1: Every multidigit absolute prime integer N is either a repunit, or can be obtained by a permutations of digits from the integer

$$B_n(a, b) = \overline{aaa \ldots ab}_{(n)} = a \cdot A_n + (b - a),$$

where a and b are different digits from the set $\{1, 3, 7, 9\}$.

Proof: Let n be the number of digits of N. We can suppose that $n > 6$. By the first three lemmas N is written by the digits $1, 3, 7, 9$ only but it does not contain in its decimal representation all of the digits $1, 3, 7, 9$, and it can contain three such digits only if N is a permutation of digits of the number $\overline{aaa \ldots abc}_{(n)}$. Let us show that this is impossible. Since N is an absolute prime, the integers

$$N_1 = \overline{a \ldots acaaaaab}_{(n)}, \qquad N_2 = \overline{a \ldots abaaaaac}_{(n)},$$

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are also absolute primes, and by Lemma 4 the integers $\overline{a \ldots ac}_{(n-6)}$ and $\overline{a \ldots ab}_{(n-6)}$ are both divisible by 7. Thus their difference, whose absolute value is $|b - c|$, is also divisible by 7, which is impossible.

Therefore either N is a repunit or else it is written by two digits only. In the latter case we need Lemma 3 again, to secure that one digit appears only once.

The prime number 7 played a significant role in the preceding considerations. But other useful primes also exist and we are going to find some of them. Note that the property of 7 most useful for us was the fact that the powers 10^i, $0 < i < 6$, had different nonzero remainders on dividing by 7. In general, by Fermat's Little Theorem for every prime $p > 5$, we have $10^{p-1} \equiv 1 \pmod{p}$.

Let $h(p)$ be the least possible positive integer such that $10^{h(p)} \equiv 1 \pmod{p}$. It is obvious that $h(p)$ is a divisor of $p - 1$ and that $10^q \equiv 1 \pmod{p}$ implies that q is divisible by $h(p)$. It is also easy to see that the powers 10^j, $0 < j < p - 1$, have different nonzero remainders on dividing by p as soon as $h(p) = p - 1$. When this is the case, 10 is said to be a **primitive root** modulo p.

Note that 10 is a primitive root modulo primes 17, 19, 23, 29 (the reader again may write a computer program to check that), but 10 is not a primitive root modulo 13 since $10^6 \equiv 1 \pmod{13}$.

Lemma 5: Let A_n be a repunit and $p > 3$ is a prime. Then $A_n \equiv 0 \pmod{p}$ if and only if $n \equiv 0 \pmod{h(p)}$.

Proof: As $10^n = 9 \cdot A_n + 1$, we have $A_n \equiv 0 \pmod{p}$ if and only if $10^n \equiv 1 \pmod{p}$ and this is equivalent to $n \equiv 0 \pmod{h(p)}$.

This simple assertion gives information about divisors of the repunits: in particular, if n is prime and $A_n = p_1 p_2 \ldots p_s$ is the factorization of A_n into prime factors, then $h(p_1) = h(p_2) = \ldots = h(p_s) = n$. For instance, $A_7 = 239 \cdot 4649$ and $h(239) = h(4649) = 7$.

Lemma 6: Let $B_n(a, b)$ be an absolute prime, p be a prime such that $n > p - 1$. Suppose that 10 is a primitive root modulo p, and a and p are relatively prime. Then n is a multiple of $p - 1$.

Proof: Let us consider the integers

$$B_i = a \cdot 10^{p-1} \cdot A_{n-p+1} + a \cdot A_{p-1} + (b - a) \cdot 10^i, \qquad 0 \le i \le p - 2,$$

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obtained from $B_n(a, b)$ by a permutation of the last $p - 1$ digits. The powers 10^i, $0 \leq i \leq p - 2$, yield all nonzero remainders on dividing by p, so do the integers $(b - a) \cdot 10^i$, $0 \leq i \leq p - 2$, and hence all the integers B_i can be simultaneously prime only in the case when the integer $L = a \cdot 10^{p-1} \cdot A_{n-p+1} + a \cdot A_{p-1}$ is divisible by p. But then, since $\mathrm{GCD}(a \cdot 10^{p-1}, p) = 1$ and $A_{p-1} \equiv 0 \pmod{p}$, it follows that A_{n-p+1} is divisible by p and by Lemma 5 n is divisible by $p - 1$.

Lemma 7: The integers $B_n(a, b)$, $a \neq b$, are not absolutely prime for $7 \leq n \leq 16$.

Proof: If $a \neq 7$, it follows from Lemma 6, applied for $p = 7$, that we need to verify the integers $B_n(a, b)$ with $n = 12$ only, whereas the case $a = 7$ requires a little bit more work. Direct calculations (or the use of a computer) here seem to be unavoidable. These calculations show that the integers $B_n(7, b)$ are either multiples of 3, or else by a permutation of digits they can be converted into multiples of 17 or 19.

Theorem 2: Let N be an absolute prime, different from repunits, that contains $n > 3$ digits in its decimal representation. Then n is a multiple of 11088.

Proof: According to the previous lemma we assume that $n > 16$. Since 10 is a primitive root modulo 17, Lemma 6 yields that n divides 16 and hence $n \geq 32$. We can repeat this argument three times, using the primes 19, 23, 29, to obtain that n is a multiple of 18, 22 and 28, respectively. Therefore n divides $\mathrm{LCM}(16, 18, 22, 28) = 11088$.

Richert [5] used in addition the primes 47, 59, 61, 97, 167, 179, 263, 383, 503, 863, 887, 983 to show that the number n of digits of the absolute prime number $B_n(a, b)$ is divisible by $321, 653, 308, 662, 329, 838, 581, 993, 760$. He also mentioned, that by using the tables of primes and their primitive roots up to 10^5, it is possible to show that $n > 6 \cdot 10^{175}$.

Let us discuss now what pairs (a, b) can appear in a decimal representation of an absolute prime $B_n(a, b)$ with $n > 3$ (if it exists at all!).

Theorem 3: If for $n > 3$ the integer $B_n(a, b)$ is an absolute prime, then $(a, b) \neq (9, 7), (9, 1), (1, 7), (7, 1), (3, 9), (9, 3)$.

Proof: Let us write down the following equality

$$9A_n - 2 \cdot 10^r = 10^n - 1 - 2 \cdot 10^r = 10^n + 1 - 2(10^r + 1).$$

We know from Theorem 2 that n must be even. Write $n = 2^m \cdot u$, where u is odd. Then for $r = 2^m$ the integer $10^n + 1$ is divisible by $10^r + 1$, and the integer $9A_n - 2 \cdot 10^r$ is composite. But this integer can be obtained by a permutation of digits of $B_n(9,7)$.

Furthermore,

$$B_n(9,1) = 9A_n - 8 = 10^n - 9 = (10^{n/2} - 3)(10^{n/2} + 3),$$

and this number is also composite.

Finally, since n by Theorem 2 is divisible by 3, the sums of the digits of $B_n(1,7)$ and $B_n(7,1)$ are also divisible by 3. Hence these numbers are composite as well as $B_n(9,3)$ and $B_n(3,9)$.

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