On Families of Solutions for Meta-Fibonacci Recursions Related to Hofstadter-Conway \$10000 Sequence

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Overview

1. Abstract
2. A brief look on the meta-Fibonacci recurrence relations
    - Birth of concept of meta-Fibonacci recurrence relations
    - Hofstadter-Conway $10000 sequence as a famous example of meta-Fibonacci recursions and its well-known curious properties
    - Common properties of solutions of meta-Fibonacci recurrence relations
3. Main difficulties of general theory of strange recursions
4. On new curious families of solutions related to Hofstadter-Conway $10000 sequence

Abstract

Hofstadter-Conway \$10000 sequence (A004001 in OEIS which is On-Line Encyclopedia of Integer Sequences) is a famous meta-Fibonacci sequence which is defined by recurrence relation as follows.

Hofstadter-Conway \$10000 sequence

$c(n) = c(c(n-1)) + c(n - c(n-1))$ with $c(1) = c(2) = 1$.

Based on its amazing fractal-like structure, in this study, we investigate certain solution families of the new meta-Fibonacci recursion families in terms of identifying natural sets of initial conditions and we prove their essential properties. Then, we will apply a series of methods to examine the behavioral changes of the solutions depending on the initial conditions selection. We will observe that there are interesting results in this work.

$$a_i(n) = n - a_i(a_i(n-i)) - a_i(n - a_i(n-i)).$$

$$a_i(n) = a_i(a_i(n-i)) + a_i(n - a_i(n-1)).$$

Birth of concept of meta-Fibonacci recurrence relations

- A part of a letter from D. R. Hofstadter to N. J. A. Sloane (1977).

Hofstadter Q-sequence

$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$ with $Q(1) = Q(2) = 1$.

Now the following sequence is a horse of an entirely nother color.

Q(1) = Q(2) = 1
Q(n) = Q(n-Q(n-1)) + Q(n-Q(n-2))

This sequence is absolutely CRAZY. I showed it to Paul Erdös, and he found it quite intriguing. It is like Fibonacci-numbers because you're always adding previous terms, but you are told how far back to count by the last two terms (instead of just adding them). There is NO rhyme nor rule to the behavior of this sequence (despite the fact that if you try the first several values at n = 3 times a power of 2, you will think you see a pattern. But it flops out there somewhere. In fact, it is not obvious by any means that Q(n) even EXISTS for all n -- yet it "clearly" does, from computing it out to 50,000 terms.

Birth of concept of meta-Fibonacci recurrence relations

- A first extensive study about famous Q-sequence by Klaus Pinn (Complexity). Since $Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$, $Q(n)$ is described as a child of its mother $(Q(n - Q(n-1)))$ and its father $(Q(n - Q(n-2)))$. This reasonable approach was the beginning of generation concept.

Order and Chaos in Hofstadter's $Q(n)$ Sequence

KLAUS PINN

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Birth of concept of meta-Fibonacci recurrence relations

- Does $Q(n)$ exist for all n? If it happens that $Q(n-1) \geq n$, then $Q(n)$ would refer to a nonpositive index and fail to exist. In the event of such happenstance, we say that the sequence dies. This is still open question! In general, one of the most notable question about a meta-Fibonacci recurrence is existence of an infinite solution sequence for given set of initial conditions.

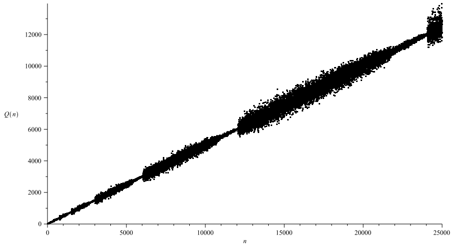

Figure: Scatterplot of Hofstadter's Q-sequence for $n \leq 25000$

Birth of concept of meta-Fibonacci recurrence relations

- After the invention of Q-sequence, in literature, there are many meta-Fibonacci sequences which are investigated by different authors. Some well-known examples as below.

Tanny sequence

$$a(n) = a(n-1-a(n-1)) + a(n-2-a(n-2)),\ a(0) = a(1) = a(2) = 1.$$

Conolly sequence

$$a(n) = a(n-a(n-1)) + a(n-1-a(n-2)),\ a(1) = a(2) = 1.$$

V-sequence

$$a(n) = a(n-a(n-1)) + a(n-a(n-4)),\ a(1) = a(2) = a(3) = a(4) = 1.$$

Hofstadter-Conway \$10000 sequence as a famous example of meta-Fibonacci recursions and its well-known curious properties

- True story of prize of Hofstadter-Conway \$10000 sequence

A004001 Hofstadter-Conway \$10000 sequence: a(n) = a(a(n-1)) + a(n-a(n-1)) with a(1) = a(2) = 1. +40
(Formerly M0276) 206

1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 8, 8, 9, 10, 11, 12, 12, 13, 14, 14, 15, 15, 15, 16, 16, 16, 16, 16, 17, 18, 19, 20, 21, 21, 22, 23, 24, 24, 25, 26, 26, 27, 27, 27, 28, 29, 29, 30, 30, 30, 31, 31, 31, 31, 32, 32, 32, 32, 32, 32, 33, 34, 35, 36, 37, 38, 38, 39, 40, 41, 42 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS On Jul 15 1988 during a colloquium talk at Bell Labs, John Conway stated that he could prove that a(n)/n -> 1/2 as n approached infinity, but that the proof was extremely difficult. He therefore offered \$100 to someone who could find an n_0 such that for all n >= n_0, we have |a(n)/n - 1/2| < 0.05, and he offered \$10,000 for the least such n_0. I took notes (a scan of my notebook pages appears below), plus the talk - like all Bell Labs Colloquia at that time - was recorded on video. John said afterwards that he meant to say \$1000, but in fact he said \$10,000. I was in the front row. The prize was claimed by Colin Mallows, who agreed not to cash the check. - N. J. A. Sloane, Oct 21 2015

Hofstadter-Conway \$10000 sequence as a famous example of meta-Fibonacci recursions and its well-known curious properties

- The name of this sequence comes from a prize of \$10000 that John H. Conway offered for a detailed analysis of this sequence. Colin Mallows provided such an analysis.

- The sequence $c(n)$ is monotone increasing with successive differences either 0 or 1. So the sequence is well-defined and $c(n)$ exists for all n.

- $\lim_{n\to\infty} \frac{c(n)}{n} = \frac{1}{2}$.

- $c(n) \geq \frac{n}{2}$ for all n, with equality if and only if n is a power of 2.

Hofstadter-Conway \$10000 sequence as a famous example of meta-Fibonacci recursions and its well-known curious properties

- Below figure illustrates Hofstadter-Conway \$10000 sequence's fractal-like structure (a curious series of increasingly greater versions of consecutive humps). Graphs of these humps converge to a special curve form that can be parametrized in terms of the Gaussian distribution, and this is proved by Kubo and Vakil.

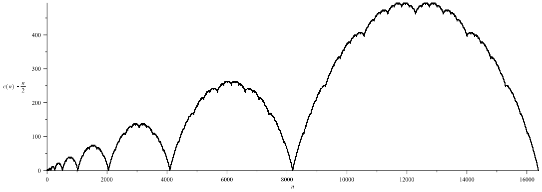

Figure: Scatterplot of $c(n) - \frac{n}{2}$ for $n \leq 2^{14}$

Hofstadter-Conway $10000 sequence as a famous example of meta-Fibonacci recursions and its well-known curious properties

- One of the most extensive study in literature about properties of Hofstadter-Conway $10000 sequence

ELSEVIER

Discrete Mathematics 152 (1996) 225–252

DISCRETE MATHEMATICS

On Conway's recursive sequence

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Received 21 June 1994

Common properties of solutions of meta-Fibonacci recurrence relations

- It is well-known that solutions of meta-Fibonacci recursions are very sensitive to selection of initial conditions.
- For some solutions, slowness is a well-known provable global property that refers successive terms of meta-Fibonacci sequence increasing by 0 or 1. Most of the time, induction is corresponding proof technique for such sequences. Nesting on components makes recursions exteremely resistant for proof attempts.
- Having an ordinary generating function is an another provable global property for another class of solutions. Quasi-periodic solutions are examples of such sequences.
- For enigmatic sequences such as Hofstadter's Q-sequence there are conjectural global properties based on generational approaches and statistical analysis, i.e. approximate self-similarity, period doubling,

Main difficulties of general theory of strange recursions

- Classification of solutions based on a selected nested recurrence is extremely difficult for many times since small change in selection of initial conditions can make an unexpected difference. So finding patterns or families of initial conditions are really difficult in order to categorize behavioral similarities, most of the time.

- Provable connections between different nested recurrence relations are extremely difficult in terms of categorizing of behavioral similarities. Again, small change in a recurrence formula (i.e., a shift parameter) can bring about dramatical difference.

- In provable statements, proofs can be extremely difficult based on induction with relatively complex inductive hypotheses. There is no easy way to generalize hypotheses for different type of recurrences. Most of the time, each recurrence necessitates its additional hypotheses.

Main difficulties of general theory of strange recursions

- The theory of strange recursions may be regarded as the discrete case of the theory of strange attractors (Solomon W. Golomb). Meta-Fibonacci is still unexplored field of nonlinear recurrences in terms of behavioral characteristics.

around 1990

213-740-8729

DISCRETE CHAOS:
SEQUENCES SATISFYING "STRANGE" RECURSIONS

213-740-7333

~~213-743-2440~~

Solomon W. Golomb

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1. Historical Summary

Analogous to the Fibonacci Sequence, defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$

Main difficulties of general theory of strange recursions

- Iterated Floor Function, Algebraic Numbers, Discrete Chaos, Beatty Subsequences, Semigroups by Aviezri S. Fraenkel.

3. Discrete chaos or discrete dynamical systems

D. Hofstadter [26] defined a sequence $\{a_n\}$ by

$$a_1 = a_2 = 1, \qquad a_m = a_{m-a_{m-1}} + a_{m-a_{m-2}} \qquad (m \geq 3),$$

which he called a "strange" recursion, since the subscripts depend on terms of the sequence itself. This and some other "strange" sequences appear to behave quite irregularly and are likely to produce good "pseudo-random" numbers, whereas other "strange" sequences are more regular. The theory of strange recursions has been called "discrete chaos" by Golomb [21]. One of the difficulties of this theory is to decide whether a strange looking sequence is indeed "crazy", or contains regular structure, if hidden. A case in point is the sequence

$$a_1 = a_2 = 1, \qquad a_m = a_{m-a_{m-1}} + a_{a_{m-1}} \qquad (m \geq 3)$$

defined by J. H. Conway, which he believed to have unpredictable convergence behavior, but Mallows [30] established enough of the structure to exhibit the asymptotic behavior.

Main difficulties of general theory of strange recursions

- Pinn studied statistically several types of meta-Fibonacci sequences which have unpredictable nature and he conclude that there is some evidence that the different sequences studied share a universality class.

A Chaotic Cousin of Conway's Recursive Sequence

Klaus Pinn

CONTENTS

I introduce the recurrence $D(n) = D(D(n-1))+D(n-1-D(n-2))$, $D(1) = D(2) = 1$, and study it by means of computer experiments. The definition of $D(n)$ has some similarity to that of Conway's sequence defined by $a(n) = a(a(n-1)) + a(n - a(n-1))$, $a(1) = a(2) = 1$. However, unlike the completely regular and predictable behaviour of $a(n)$, the D-numbers exhibit chaotic patterns. In its statistical properties, the D-sequence shows striking similarities with Hofstadter's Q(n)-sequence, given by $Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$, $Q(1) = Q(2) = 1$. Compared to the Hofstadter sequence, D shows higher structural order. It is organized in well-defined "generations", separated by smooth and predictable regions. The article is complemented by a study of two further recurrence relations with definitions similar to those of the Q-numbers. There is some evidence that the different sequences studied share a universality class.

Main difficulties of general theory of strange recursions

- A part of An Invitation to Nested Recurrence Relations by Steve Tanny in Canadam 2013. Finding closed form solutions is another challenge!

$$R(n) = R(n-R(n-1))+1,\ R(1) = 1$$

(Golomb)

Early recursion, closed form solution: R(n) = fl{[1+√(8n)]/2}.

Solution: 1,2,2,3,3,3,4,4,4,4,...; each positive n appears n times. Sequence is slow. First proof by induction.

"This furnishes an important example of a recursion which looks as "strange" as several others that we have considered, but where the resulting sequence is completely regular and predictable. It is a challenging unsolved problem to categorize those "strange" recursions which have well-behaved, closed-form solutions." (Golomb, ca. 1986?) – **Still true today!!**

Main difficulties of general theory of strange recursions

- Connections between different nested recurrence relations and solutions of them are highly difficult to detect.

Hindawi
Complexity
Volume 2017, Article ID 2614163, 8 pages
https://doi.org/10.1155/2017/2614163

Research Article

On Hofstadter Heart Sequences

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Main difficulties of general theory of strange recursions

- Generalizations with high behavioral similarities are difficult to detect, especially if the sequences are not slow or quasi-periodic. Theoretically, there are infinite number of initial condition combinations in order to discover new solution sequences. So the classification of them is a problematic issue.

Hindawi
Complexity
Volume 2018, Article ID 8517125, 8 pages
https://doi.org/10.1155/2018/8517125

WILEY | Hindawi

Research Article

On a Generalization of Hofstadter's Q-Sequence: A Family of Chaotic Generational Structures

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On new curious families of solutions related to Hofstadter-Conway $10000 sequence

- In order to find new curious solutions related to Hofstadter-Conway $10000 sequence, we will mainly use the method that introduced by work below.

Open Math. 2018; 16: 1490–1500 DE GRUYTER

Open Mathematics

Research Article

Altug Alkan*

On a conjecture about generalized Q-recurrence

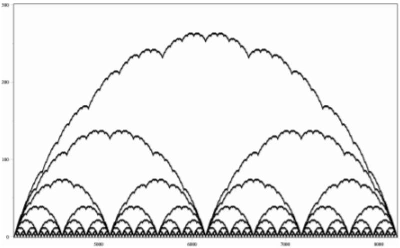
https://doi.org/10.1515/math-2018-0124

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

- Corresponding method simply depends on fractal-like collection of fractal-like sequences as below. Although proofs of all related facts about this methodolgy is too long for this presentation, we will simply mention about general framework in this work.

1492 — A. Alkan **DE GRUYTER**

Fig. 1. $a_{2^t}(n) - \frac{n}{2}$ for $2^{12} \le n \le 2^{13}$ and $0 \le t \le 7$, respectively.



On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

- How do we demonstrate that there are infinitely many theoretical solutions with successive terms increasing by 0 or 1 such as Hofstadter-Conway \$10000 sequence? If the proofs are possible, are they meaningful solutions that have unexpected results in terms of behavioral characteristics? We will try to find answers to these questions.

$$a_i(n) = n - a_i(a_i(n-i)) - a_i(n - a_i(n-i)).$$

$$a_i(n) = a_i(a_i(n-i)) + a_i(n - a_i(n-1)).$$

- We will apply approach of previous study for the following recurrences in order to explore new results.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

- We determine the asymptotic properties of our recurrences for natural cases based on Hofstadter-Conway \$10000 sequence. This part contains several types of number theoretical works.

- We propose the initial condition functions based on result of asymptotic properties of our recurrences. These functions are infinite integer sequences which we can parameterize how many terms we can get as the initial condition. These initial condition families that corresponding function covers are our potential source to discover infinitely many different solutions.

- We determine the changing points on corresponding ceiling or floor function based on constructing of related propositions. In other words, we prove the minumum distances between beginning points of different solutions on initial condition functions.

- Then we construct complex inductive hypothesis in order to prove existence of theoretical solutions. We mainly focus on slow solutions due to nature of our recurrences in this study. If one can completely automate such methodology, we believe that this would be an important contribution in terms of general theory of these strange recursions.
- Then we investigate nature of strange recursion in terms of unit change in initial conditions. We believe that it would be nice if we can model the behaviour of strange recursions according to change of initial conditions.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

Theorem

Let $a_i(n) = n - a_i(a_i(n-i)) - a_i(n - a_i(n-i))$ *for* $n > 2 \cdot \lceil \frac{i}{2} \rceil$*, with the initial conditions* $a_i(n) = \lceil \frac{n}{2} \rceil$ *for* $1 \leq n \leq 2 \cdot \lceil \frac{i}{2} \rceil$*.* $a_i(n+1) - a_i(n) \in \{0, 1\}$ *for all* $n, i \geq 1$ *and* $a_i(n)$ *hits every positive integer.*

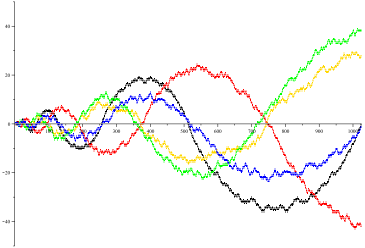


Figure: $s_1(n)$: black, $s_2(n)$: red, $s_3(n)$: yellow, $s_4(n)$: green, $s_5(n)$: blue where $s_i(n) = a_i(n) - \frac{n}{2}$ for $n \leq 2^{10}$.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

- A nice connection beetween $a_1(n)$ and $c(n)$ which is Hofstadter-Conway \$10000 sequence. They behave in a complete relationship. A new auxilary method shows the connection in their generational structures from another perspective. Both sequences have same generational boundaries. See next slide for main idea of corresponding method and intersecting behaviors.

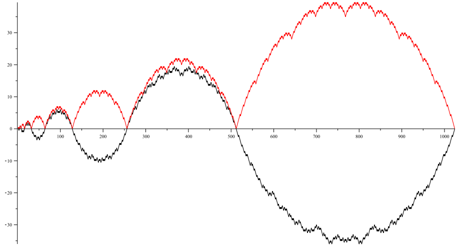


Figure: Scatterplots of $c(n) - \frac{n}{2}$ (red) and $a_1(n) - \frac{n}{2}$ (black) for $n \leq 2^{10}$.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

Let $b(n) = n - b(c(n)) - b(n - c(n))$ and
$t(n) = n - t(a_1(n)) - t(n - a_1(n))$ with $b(1) = b(2) = t(1) = t(2) = 1$.

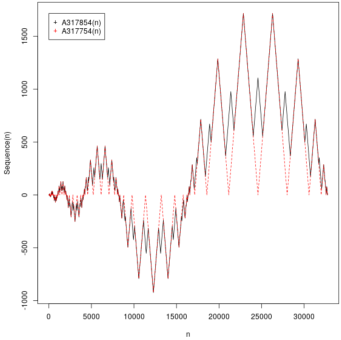


Figure: Plots of $2 \cdot b(n) - n$ which is A317754 in OEIS (red) and $2 \cdot t(n) - n$ which is A317854 in OEIS (black) for $n \leq 2^{15}$.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

Theorem

Let $a_i^*(n) = n - a_i^*(a_i^*(n-1)) - a_i^*(n - a_i^*(n-1))$ *for* $n > 2^i$*, with the initial conditions* $a_i^*(n) = \lceil \frac{n}{2} \rceil$ *for* $1 \leq n \leq 2^i$*.* $a_i^*(n+1) - a_i^*(n) \in \{0, 1\}$ *for all* $n, i \geq 1$ *and* $a_i^*(n)$ *hits every positive integer.*

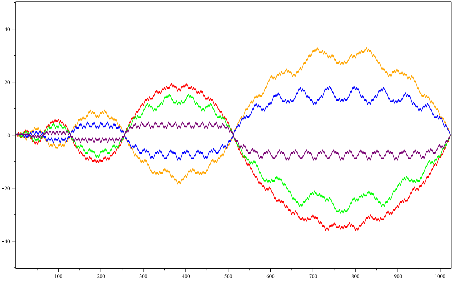


Figure: $s_1^*(n)$: red, $s_2^*(n)$: yellow, $s_3^*(n)$: green, $s_4^*(n)$: blue, $s_5^*(n)$: purple where $s_i^*(n) = a_i^*(n) - \frac{n}{2}$ for $n \leq 2^{10}$.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

Theorem

Let $a_{i,j}(n) = a_{i,j}(a_{i,j}(n-j)) + a_{i,j}(n - a_{i,j}(n-1))$ *for* $n > 2 \cdot (i-1) + 3 \cdot \lfloor \phi \cdot i \rfloor$, *with the initial conditions* $a_{i,j}(n) = \left\lfloor \frac{n+2}{1+\phi} \right\rfloor$ *for* $1 \leq n \leq 2 \cdot (i-1) + 3 \cdot \lfloor \phi \cdot i \rfloor$ *where* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $j \in \{1, 2\}$. $a_{i,j}(n+1) - a_{i,j}(n) \in \{0, 1\}$ *for all* $n, i \geq 1$ *and* $a_{i,j}(n)$ *hits every positive integer.*

On new curious families of solutions related to Hofstadter-Conway $10000 sequence

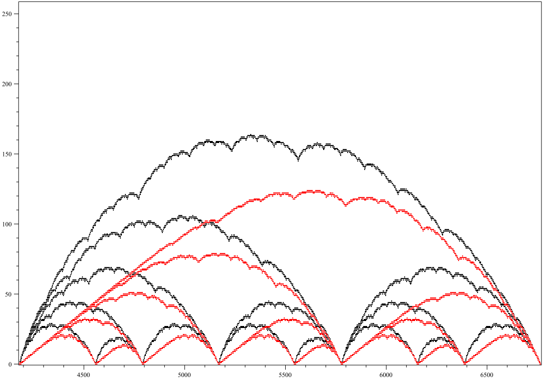


Figure: Plots of $a_{i,1}(n) - \left\lfloor \frac{n+2}{1+\phi} \right\rfloor$ (black) and $a_{i,2}(n) - \left\lfloor \frac{n+2}{1+\phi} \right\rfloor$ (red) for $F(19) \leq n \leq F(20)$ and $1 \leq i \leq 8$.

On new curious families of solutions related to Hofstadter-Conway \$10000 sequence

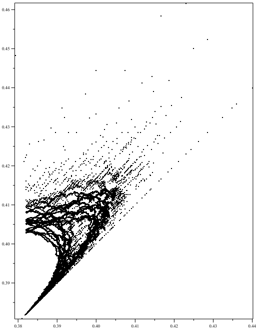


Figure: Plot of $\frac{a_{2,1}(n)}{n}, \frac{a_{1,1}(n)}{n}$ for $F(8) \leq n \leq F(27)$. A scatterplot showing how changing initial conditions (as we determined with our methods) affect asymptotic behavior.

On new curious families of solutions related to Hofstadter-Conway $10000 sequence

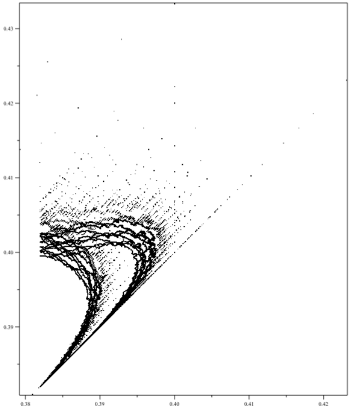


Figure: Plot of $\frac{a_{2,2}(n)}{n}, \frac{a_{1,2}(n)}{n}$ for $F(8) \leq n \leq F(27)$. A similar scatterplot showing how changing initial conditions (as we determined with our methods) affect asymptotic behavior.

Conclusion

We apply our methodology for two strange meta-Fibonacci recursions related to Hofstadter-Conway $10000 sequence. We construct infinitely many theoretical solutions with curious generational structures and we also observe interesting results. While this methodology provides meaningful results in order to overcome some difficulties of analysis of meta-Fibonacci recursions which is intriguing subfamily of nonlinear recurrences, we hope that these attempts will be meaningful for future communications in this research field.

Thank You!