which relates changes in Z_0 to mesh impedance changes and mesh currents ... two conjugate networks. The proof is completed most simply by ...sing meshes such that only one independent mesh current flows through the variable element $Z^{(n)}$. Equation (13) then becomes

$$I_0^2 \delta Z_0 = I^{(n)\prime} \delta Z^{(n)} I^{(n)} \tag{14}$$

which gives the required extension of Vratsanos' theorem,

$$\partial Z_0 / \partial Z^{(n)} = I^{(n)} I^{(n)\prime} / I_0^2. \tag{15}$$

The trivial case where the terminals of $Z^{(n)}$ are not both connected is also covered by (15) since the element current is zero. If the two-port obtained by removing $Z^{(n)}$ is reciprocal

$$I^{(n)} = I^{(n)\prime} \tag{16}$$

and Vratsanos' theorem (1) is recovered. That the element currents of the two conjugate networks are involved symmetrically in (15) is not surprising in view of (7), which says that Z_0 and hence the dependence of Z_0 on $Z^{(n)}$, are the same for both networks.

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On Biquadratic Impedances with Two Reactive Elements

The paper by Foster [1] has rekindled considerable interest in the synthesis of biquadratic impedance functions with the least number of elements (see [2], and references). The difficulty of the problem has forced most authors to adopt a "census" approach. The networks realizing biquadratic impedance functions are divided into subclasses according to the total number of *RLC* elements and/or the number of reactive elements. Realizability conditions for each subclass are then investigated. In this correspondence, we point out that one subclass of networks, widely believed to be completely settled, actually has the status of an unsolved problem, and therefore requires further research.

In a recent article by Vasiliu [2], we read the following statement, "The impedance of any network consisting of four resistors and two reactive elements is always realizable by a two-reactive five-element network [1]." Now, in this statement, if "any network" means "any series-parallel network," then the statement is true. However, a proof is not to be found in its quoted reference [1]. In fact, for series parallel networks, the statement is true even if there are more than four resistors. A rigorous proof may be found in [3].

On the other hand, if "any network" means "any network, whether series-parallel or not," then the validity of the statement has never been established in published literature [5]. The popular belief that the statement is true for nonseries-parallel networks probably stems from one sentence in [1], "In addition, it may be pointed out that nothing is gained by adding resistors in excess of the three called for in the original census."

This turns out to be quite difficult to prove. The census approach is no ...nger applicable since the number of resistors can be any positive integer. ...e level of difficulty of the problem, as well as one possible approach to its solution, has been expressed by Auth in the conclusion of his paper [4].

Manuscript received April 23, 1970; revised June 18, 1970.

To conclude, we restate the conjecture with the hope of stimulating further research and obtaining an answer.

Conjecture: the impedance function of any one-port network consisting of one capacitance, one inductance, and any number of resistances, is always realizable with at most one capacitance, one inductance, and three resistances.

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On the Biplanar Crossing Number

Abstract—In the design of printed and integrated circuits it is desirable to minimize the number of jumpers and via-holes. Linear graph theory provides apparatus useful in the formal specification and solution of this problem.

Prior work has concentrated on the drawing of graphs in a single plane. This correspondence, however, is concerned with determining the minimum number of crossings for two subgraphs of a graph whose union is the entire graph. The two subgraphs correspond to the two sides of a printed circuit board or to two layers of metalization on an integrated circuit. An upper bound on this crossing number and a model yielding this bound have been found for complete graphs.

I. INTRODUCTION

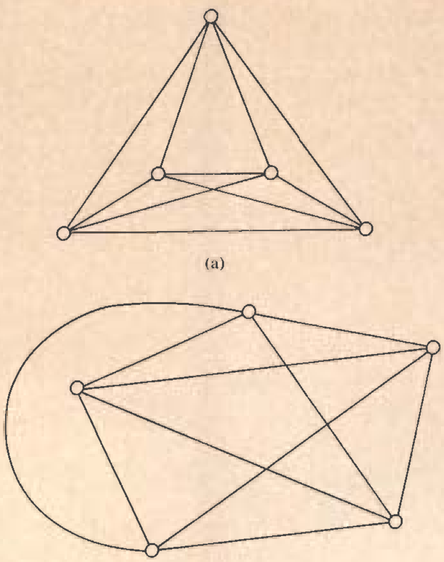
A major problem in computer-aided design of printed and integrated circuits is to place the component interconnections so as to minimize the number of crossings and hence reduce the number of via-holes or jumpers. Formally replacing the components by vertices and the connections by edges transforms a printed circuit to a linear graph. The graph problem is to find a drawing of the graph in a chosen surface so that the edges have the fewest possible number of crossings. The contributions of graph theory to the crossing minimization problem may be divided into stages. The first stage is to determine whether or not a graph can be drawn in the plane without crossings, i.e., whether or not the graph is planar. Kuratowski [1] in 1937 showed that a graph is planar if and only if it does not contain a subgraph homeomorphic to either K_5, the complete graph on five vertices, or to $K_{3,3}$, the complete bipartite graph. Two graphs are homeomorphic if they differ only by vertices of incidence two. Fig. 1 shows two graphs homeomorphic to K_5.

MacLane [2] showed that a graph is planar if and only if it contains a complete set of cycles such that no edge is in more than two of the cycles. Several investigations have given computerizable algorithms which test graphs for planarity. For examples, see Goldstein [3], Tutte [4], Dunn and Chan [5], Fisher and Wing [6], and Lempel *et al.* [7].

The second stage is to determine the thickness t or the fewest number of planar subgraphs whose union is the whole graph. Given t and the subgraphs, it is possible to construct the corresponding printed circuit on t layers without crossings. For early graph results in this stage, see Beineke *et al.* [8], [9] and Ringel [10].

The third stage is to draw the graph G in exactly one plane so that the edges have the fewest number Cr_1 (G) of crossings. Conjectures for $Cr_1(K_n)$ and $Cr_1(K_{m,n})$ can be found in Guy [11], [12] and Zarankiewicz [13],

Manuscript received September 12, 1969; revised July 20, 1970. This research was supported by the U. S. Naval Research Laboratory under Project RR003-02-41-6150.



(a)

(b)

Fig. 1. (a) K_5 with one crossing. (b) K_5 with three crossings.

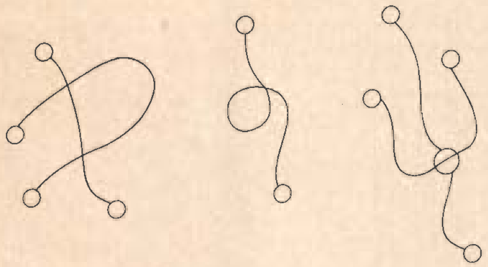


Fig. 2. Forbidden crossings.

respectively. Lerda and Majorani [14] and Nicholson [15] have considered the crossing problem for arbitrary graphs in the plane.

The fourth stage, considered in this correspondence, is to determine the fewest number of edge crossings when the graph is drawn on two planes. Since most printed circuit boards have two sides, it is clear that the most efficient constructs of printed circuits have connections on both sides.

II. Biplanar Models

Any graph may be drawn in the plane if the edges are permitted to cross. For example, the graph K_5 which has five vertices with each vertex joined to every other, may be drawn in the plane with one crossing (Fig. 1(a)) or with three crossings (Fig. 1(b)).

In this correspondence, crossings are not permitted at vertices and no two edges may cross more than once.[1] Fig. 2 shows the forbidden crossings.

Guy [12] conjectures that the minimum number of crossings for any drawing of K_n, the complete graph on n vertices, in the plane is

$$\widetilde{Cr}_1(K_n) = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$$

where $[x]$ denotes the greatest integer not exceeding x. Moreover, Zarankiewicz [13] conjectured that the fewest number of crossings for any drawing in the plane of the complete bipartite graph $K_{m,n}$ is

$$\widetilde{Cr}_1(K_{m,n}) = \left[\frac{n}{2}\right]\left[\frac{m}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{m-1}{2}\right].$$

[1] No allowances have been made for cases where the insulating properties of a component surface allow crossings through a component (or vertex) as with thin-film RC circuits.

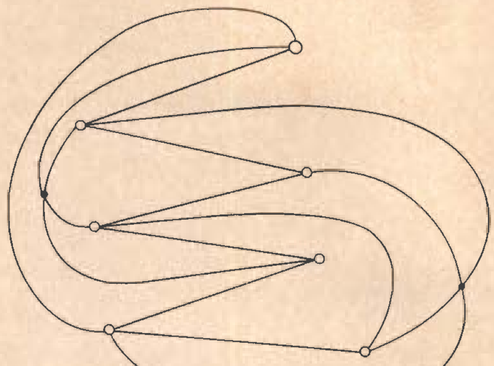
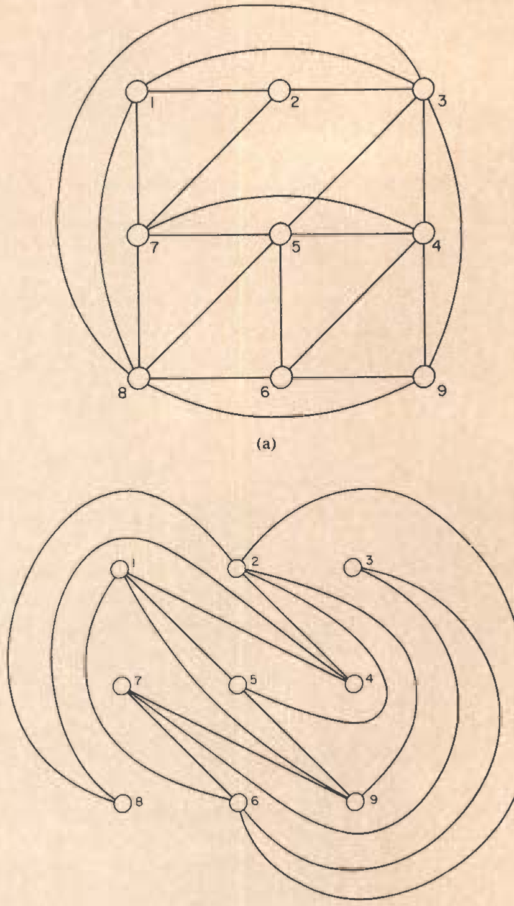


Fig. 3. $K_{3,4}$ with two crossings.



(a)

(b)

Fig. 4. Biplane drawing of K_9 with one crossing.

According to this result $K_{3,4}$ can be drawn in the plane with two crossings. Fig. 3 shows such a drawing.

The complete graph on nine vertices K_9, is interesting since it is the smallest complete graph which cannot be drawn on two planes without crossings. Hence the printed circuit corresponding to K_9 cannot be imbedded on the two sides of one printed circuit card without via-holes. Fig. 4 shows a biplane drawing of K_9 with just one crossing.

The crossing number $Cr_1(G)$ of a graph G is the fewest number of crossings needed to draw the graph in the plane. The biplanar crossing number $Cr_2(G)$ is defined as the fewest number of crossings needed in order to draw the graph in two planes. Similarly, $Cr_n(G)$ is the n-planar crossing number and is the fewest number of crossings for any drawing of G in

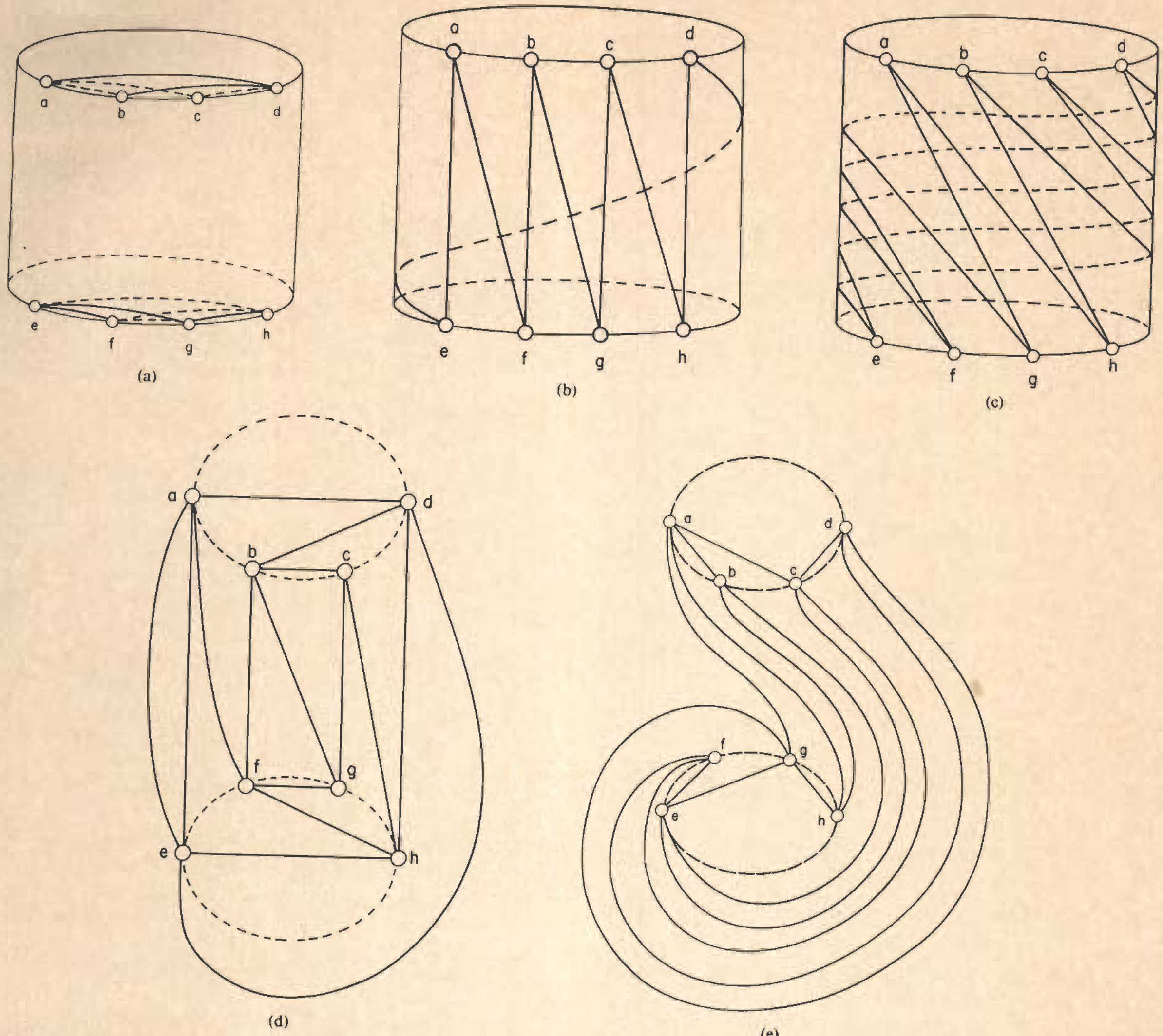


Fig. 5. (a) Top and bottom of cylinder model for K_8. (b) Outside lateral surface of cylinder model for K_8. (c) Inside lateral surface of cylinder model for K_8. (d) Outside top, bottom, and lateral surfaces of cylinder model drawn on one side of plane. (e) Inside top, bottom, and lateral surfaces of cylinder model drawn on other side of plane.

n planes. By definition

$$Cr_2(G) = \min_{H \cup K} (Cr_1(H) + Cr_1(K))$$

where H and K range over the disjoint subgraphs of G whose union is G. Thus in order to estimate $Cr_2(K_n)$ it is only necessary to give a model of K_n where a particular H and K are specified. A simple biplanar model for K_n is obtained by decomposing K_n into three subgraphs

$$K_{[n/2]}; \qquad K_{\{n/2\}}; \qquad K_{[n/2],\{n/2\}}.$$

The complete graphs on $[n/2]$ and $\{n/2\}$ vertices form H and the complete bipartite graph on $[n/2]$, $\{n/2\}$ vertices forms K. Here $\{x\}$ denotes the least integer not less than x. Drawing $K_{[n/2]}$ and $K_{\{n/2\}}$ on one side of the plane and $K_{[n/2],\{n/2\}}$ on the other side gives a model for K_n with

$$Cr_1(K_{[n/2]}) + Cr_1(K_{\{n/2\}}) + Cr_1(K_{[n/2],\{n/2\}})$$

crossings. Each of the Cr_1 numbers can be closely estimated by the previ[illegible]y given conjectures. If n is an even integer, i.e., $n = 2r$, then

$$Cr_2(K_n) = Cr_2(K_{2r}) < 2Cr_1(K_r) + Cr_1(K_{r,r})$$

$$\leq \frac{1}{2}\left[\frac{r}{2}\right]\left[\frac{r-1}{2}\right]\left[\frac{r-2}{2}\right]\left[\frac{r-3}{2}\right] + \left[\frac{r}{2}\right]^2\left[\frac{r-1}{2}\right]^2.$$

If the maximal complete bipartite subgraphs of a graph are known, then this decomposition technique yields a quick estimate to the biplanar crossing number.

A closer upper bound for $Cr_2(K_n)$ is obtained from the following model illustrated in Fig. 5 for K_8 which includes a model given by Blažek and Koman [16] for the planar crossing number. Suppose n is an even integer. Place the vertices of the graph $K_{n/2}$, the complete graph on $n/2$ vertices, on a circle and place the edges of $K_{n/2}$ inside the circle of either side of the plane so as to obtain the fewest number of crossings. This can be done with $\tilde{C}r_1(K_{n/2})$ crossings by use of the following model (see [16]). Let the vertices of the graph $v_1, \cdots, v_l (l = n/2)$ be the l vertices of a regular l-sided convex polygon. Construct two sets of edges for K_l. The first set consists of all diagonals paralled to $v_{i-1}v_{i+1}$ for $i = 1, 2, \cdots, [\frac{1}{4}(n+3)]$ and to $v_i v_{i+1}$ for $i = 1, 2, \cdots, [\frac{1}{4}(n+1)]$. The second set consists of the remaining diagonals. This second set is mapped stereographically onto the exterior of the polygon. This model of $K_l (l = n/2)$ has a Hamiltonian path whose edges have no crossings and whose total number of crossings equals $\tilde{C}r_1(K_{n/2})$. In this model identify the Hamiltonian path with the boundary of the circle and identify the interior and exterior with the two sides of the plane interior to the circle. Fix copies of this particular drawing of $K_{n/2}$ to the top and bottom of a cylinder so that the top vertices line up with the bottom vertices. On the sides of the cylinder run counterclockwise geodesics from each of the $n/2$ top vertices to the $n/2$ bottom vertices always placing the first $[n/4]$ edges from each vertex on the outside surface and the remaining on the

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TABLE I

n	5	6	7	8	9	10	11	12	13	14	15	16	17
$\hat{C}r_2(K_n)$	0	0	0	0	4	7	12	18	37	53	75	100	152
$\tilde{C}r_1(K_n)$	1	3	9	18	36	60	100	150	225	315	441	588	784

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inside surface. If n is odd, do the preceding construction for $(n+1)$ and then remove one vertex and all the edges abutting on it.

The number of crossings obtained on the sides of the cylinder is equal to

$$\frac{n^2(n-4)(n-8)}{384}, \qquad \text{when } n = 4k$$

$$\frac{n(n-2)(n-4)(n-6)}{384}, \qquad \text{when } n = 4k+2$$

$$\frac{(n-1)(n-3)^2(n-5)}{384}, \qquad \text{when } n = 4k+1$$

$$\frac{(n+1)(n-3)^2(n-7)}{384}, \qquad \text{when } n = 4k+3.$$

Adding to these the number of crossings from the top and bottom, i.e., $2\tilde{C}r_1(K_{n/2})$ gives

$$\frac{k(k-1)(k-2)(7k-3)}{6}, \qquad \text{when } n = 4k$$

$$\frac{k(k-1)(7k^2-3k-1)}{6}, \qquad \text{when } n = 4k+2$$

$$\frac{k(k-1)(7k^2-10k+4)}{6}, \qquad \text{when } n = 4k+1$$

$$\frac{k^2(k-1)(7k+4)}{6}, \qquad \text{when } n = 4k+3.$$

Denote by $\hat{C}r_2(K_n)$ the number of crossings in this model. Now $Cr_2(K_n) \leq \hat{C}r_2(K_n)$ since $Cr_2(K_n)$ is the fewest number of crossings obtained from all decompositions of K_n. Equality does not hold since $C_2(K_9)=1$ and $\hat{C}r_2(K_9)=4$. However, Table I shows the improvement of $\hat{C}r_2(K_n)$ over $\tilde{C}r_1(K_n)$, the conjectured value for $Cr_1(K_n)$.

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Number of Spanning Trees in a Wheel

Abstract—A recurrence relation for the number of spanning trees $f(n)$ in the wheel W_n, where $n \geq 3$, is obtained as $f(n+1)-f(n)=L_{2n+1}$, where $f(3)=16$ and where L_k is the kth number in the Lucas series 1, 3, 4, 7, $\cdots$, L_k, $\cdots$, where $L_k=L_{k+1}\ L_{k-1}$ for $k>1$. Alternately, $f(n) = L_n^2 - 4\delta$ where $\delta=0$ for n odd and 1 for n even, thus confirming $f(n)$ as a square number for n odd and serving to verify a previous finding in 1969 by Sedlacek that $f(n)=((3+\sqrt{5})^n+(3-\sqrt{5})^n)/2^n-2$.

Introduction

The wheel W_n of order n, where $n \geq 3$, is defined by its inventor Tutte [1] as the graph obtained from an n-gon P_n by adjoining one new vertex h and n new links joining h to the n vertices of P_n. P_n is called the rim and h the hub of W_n. The edges incident with h are the spokes. In this correspondence, W_n is labeled so that the spoke s_i is incident with the vertex v_i in P_n and the edge r_i in P_n is incident with the vertices v_i, v_{i+1}, $i=1, 2, \cdots, n (v_{n+1} \triangleq v_1)$. The labeled wheel of order 6 is shown in Fig. 1.

The wheel W_n represents an important class in both the theory and application of planar graphs. One such application is Benedict's study of self-dual electric networks [2], concerned in part with self-dual graphs each graph G of which has $c(G)$ spanning trees, where $c(G)$ is a perfect square. In their pioneering work on self-dual graphs, Smith and Tutte [3], [4] revealed that self-dual graphs obtained by reflection in the center of the sphere have this property, and that W_n, n odd, is a special case of these [5]. Benedict [6] in turn suggested that it might therefore be interesting to derive $c(W_n)=f(n)$ for all n. The solution reported herein was duly obtained. One of the reviewers of the first draft of this article (for whose advice the author is most appreciative), however, cited a prior solution by Sedlacek [7]. The two solutions are quite different in form, as it happens, and are derived from quite different approaches.

Spanning Tree Enumeration in W_n

Contracting the rim edge r_n in W_n to a new vertex w_1, then relabeling w_1 as v_1 and the edge s_n as t_n, results in the multigraph $W_{n-1}+t_n$ when $n>3$, illustrated in Fig. 2 for $n=6$ and in the multigraph R_2+t_3 of Fig. 3 when $n=3$. By a well-known rule in tree enumeration, Seshu and Reed [8], the number of spanning trees in W_n is

$$\begin{aligned} f(n) &= c(W_{n-1}+t_n) + c(W_n - r_n), \qquad \text{for } n > 3 \\ &= c(R_2+t_3) + c(W_3 - r_3), \qquad \text{for } n = 3 \end{aligned} \tag{1}$$

where $W_n - r_n$ is the graph obtained on removing the rim edge r_n from W_n. The number $c(R_2+t_3)$ in (1) is 8, and the graph $W_n - r_n$ is isomorphic to the terminated simple ladder L_{n-1} of order $n-1$ [9]. L_{n-1} has

$$c(L_{n-1}) = u_{2n}, \qquad \text{for } n \geq 1 \tag{2}$$

where $u_k = u_{k+1} - u_{k-1}$, $k>0$, is the $(k+1)$th number in the basic Fibonacci series

$$u_0, u_1, u_2, \cdots, u_k, \cdots = 0, 1, 1, \cdots, u_k, \cdots. \tag{3}$$

Thus, in (1)

$$c(W_n - r_n) = u_{2n}, \qquad \text{for } n \geq 3. \tag{4}$$

For example, there are 144 spanning trees in $W_6 - r_6$.

The number $c(W_{n+1}+t_n)$, $n>3$, in (1) is given by

$$c(W_{n-1}+t_n) = c(W_{n-1}) + c(Q_{n-3}) \tag{5}$$

Manuscript received April 14, 1970; revised July 9, 1970.

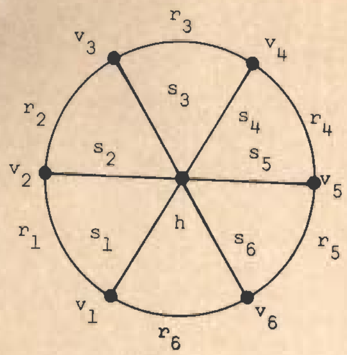

Fig. 1. W_6.

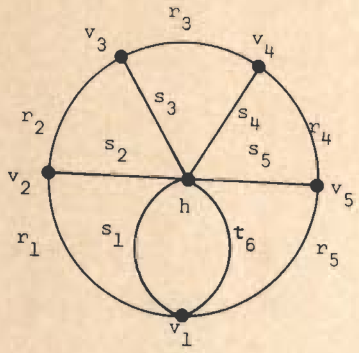

Fig. 2. $W_5 + W_6$.

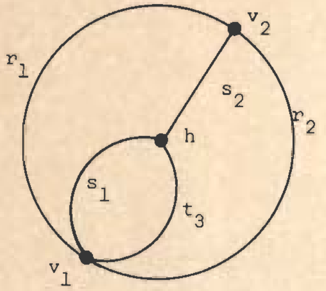

Fig. 3. $R_2 + t_3$.

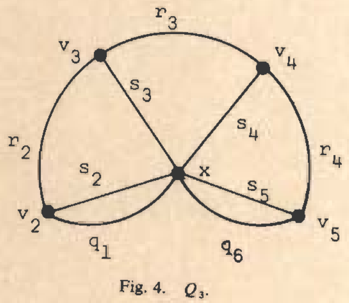

Fig. 4. Q_3.

where Q_{n-3} is the multigraph which results on contracting the spoke s_1 in W_{n-1} to a new vertex x, then relabeling the edges r_1, r_{n-1} as q_1, q_{n-1}, respectively, and where $c(W_{n-1}) = f(n-1)$. As illustrated by Q_3 in Fig. 4, Q_{n-3} is isomorphic to a multiply-terminated simple ladder of order $n-3$, the subgraph $Q_{n-3} - q_{n-1} - q_1$ being isomorphic to L_{n-3}. Thus [9]

$$c(Q_{n-3}) = u_{2n-2}. \tag{6}$$

For example, there are 55 trees in Q_3.

TABLE I

n	$f_1(n)$	$f_2(n)$	$f(n-1)$	$f(n)$
3	4	4		16
4	5	9	16	45
5	11	11	45	121
6	16	20	121	320
7	29	29	320	841
8	45	49	841	2205
9	76	76	2205	5776
10	121	125	5776	15125

Equations (1) and (4)–(6) combine to yield the recurrence relation

$$\begin{aligned} f(n+1) - f(n) &= u_{2n+2} + u_{2n}, \qquad \text{for } n \geq 3 \\ f(3) &= 16 \end{aligned} \tag{7}$$

for the number of spanning trees in W_n. Since $u_{2n+2} + u_{2n}$ is the $(2n+1)$th number in the Lucas series [10] 1, 3, 4, 7, $\cdots$, L_k, $\cdots$, where $L_k = L_{k+1} - L_{k-1}$, for $k > 1$,

$$\begin{aligned} f(n+1) - f(n) &= L_{2n+1} \\ f(3) &= 16. \end{aligned} \tag{8}$$

The values of $f(n-1)$ and $f(n)$ for $n = 3, 4, \cdots, 10$, excluding $f(2)$, are listed in Table I.

It is observed that

$$f(n) = L_n^2 - 4\delta \tag{9}$$

where $\delta = 0$ for n odd, 1 for n even; which confirms the known fact [3]–[5] that the number of spanning trees in an odd wheel is a perfect square. It is apparent from (9) that $f(n)$ is the product of two numbers

$$f(n) = f_1(n) f_2(n) \tag{10}$$

where

$$\begin{aligned} f_1(n) &= L_n - 2u_\delta \\ f_2(n) &= L_n + 2u_\delta. \end{aligned}$$

These factors are included in Table I. From the Lucas number identity [10]

$$L_n^2 = L_{2n} + 2(-1)^n \tag{11}$$

it follows immediately from (9) that

$$f(n) = L_{2n} + 2(-1)^n - 4\delta \tag{12}$$

where $\delta = 0$, 1 according as n is odd, even.

Sedlacek's [7] prior solution for $f(n)$, obtained from the determinantal formula for W_n, is

$$f(n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2. \tag{13}$$

The equality of (9) and (13) establishes the interesting Lucas numerical number identity

$$L_n^2 + 2 = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n + 4\delta \tag{14}$$

($\delta = 0$, 1 according as n is odd, even).

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Comments on "Topological Formulas for General Linear Networks"

It is difficult to avoid an occasional inadvertent repetition and unaccredited publication of prior work. Jong and Zobrist, in the above paper,[1] follow, however, a most unusual path, expressly dismissing by implication my prior work [2] as relatively uninteresting in the introduction to their paper, whose aim is the reestablishment of these results.

Thus, Jong and Zobrist's Section III on voltage-controlled sources, (4) and (5), corresponds to the "A rules" in [2] (and to lines 4 and 5 of Table I in [3]), their Section IV-A on voltage-controlled sources corresponds to my "C rules" (and to lines 2 and 3 of Table I), and their results regarding transformers are included in the "rules for transformers" [2] (and in line 11 of Table I), except that reference [2] is considerably more explicit and contains, in particular, the complete topological interpretation of terms and of their signs, further reaching results on transformers and their topological meaning, as well as many other useful conclusions. (Reference [2] is not explicitly concerned with gyrators.)

Topological rules can be expressed in several ways whose equivalence is immediately evident. In the context of such rules any two configurations are equivalent if they are composed of the same set of branches. If no transformations are used on the graph representing an electrical network, the proper configurations are "loopwoods," which comprise k-trees and sets of directed loops as special cases, and [2] is formulated in terms of these. Through obvious transformations effected by merging or grounding nodes or removing branches, a graph N can be transformed into N' such that a loopwood in N becomes a k-tree in N'. In fact, in my joint expository paper [3] formulation is in terms of k-trees, as it is in the Jong and Zobrist paper.

Jong's paper on formulas for networks containing operational amplifiers [1], while referring to some of my papers on matrix analysis of networks [3] [5], omits to mention my topological papers which contain these results (in rule 7 of [2] and in line 3 of Table I [3].) Here again the prior work immediately yields the correct signs of terms and provides a complete topological interpretation of terms.

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Manuscript received June 3, 1970.
[1] M. T. Jong and G. W. Zobrist, *IEEE Trans. Circuit Theory*, vol. CT-15, Sept. 1968, pp. 251–259.

Reply by M. T. Jong[2]

In the introduction to the paper,[1] the sentence, "Of particular interest are the methods developed by Coates and Mayeda," does not mean to imply that Dr. Nathan's prior work [2], [3] is "relatively uninteresting." The Coates–Mayeda method is explicitly mentioned, because the development in our paper is based on it.

The work in our paper took a completely different approach to developing topological formulas for networks containing any type of dependent sources, gyrators, and transformers (both practical and ideal). In employing these formulas the subnetwork consisting of the active and/or nonreciprocal elements is separated from the passive reciprocal subnetwork, and by segregating the vertices according to the formulas, only k-trees of the passive subnetwork need be found. The rules in [2], [3] apply only to voltage-constrained networks, including transformers, but not current-controlled sources, where the admittance functions are expressed in terms of trees, loops, and paths of the overall network graph in which each constraint is replaced by one or more branches. In the case of ideal transformer with floating terminals, it is replaced by six branches in lines 1 and 2 of Table III in [2]. Thus, the approach is different and the rules are not the same.

This author agrees that the terms in [2], [3] can be expressed in terms of k-trees of the passive reciprocal subnetwork through some transformations. But the results are not explicitly in terms of k-trees, as in our paper and in [1].

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[2] Manuscript received June 15, 1970.

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Sensitivity Invariants for Scattering Matrices

I. INTRODUCTION

Recent papers [1]–[3] have indicated renewed interest in the study of sensitivity invariants of networks. However, none of these seems to have treated sensitivity invariants of scattering parameters and none seems to have considered lumped–distributed networks.

In this correspondence sensitivity invariants are derived for scattering matrices of linear time-invariant networks η which are composed of lumped resistors, inductors and capacitors, gyrators, uniform transmission lines, uniformly distributed $\overline{RC}$ lines and either current-controlled voltage sources or voltage-controlled current sources. On an impedance basis the element variables are defined as follows: lumped resistors, $R_i = 1/G_i$, Ω; lumped inductors, $L_i = 1/\Gamma_i$ H; lumped capacitors, $D_i = 1/C_i$ F^{-1}; gyrators, gyrator ratio $\alpha_i = -1/\gamma_i$ Ω; characteristic impedances of uniform transmission lines, $Z_{0i} = 1/Y_{0i}$ Ω; total series resistance and total shunt capacitance of uniformly distributed $\overline{RC}$ lines, $R_{\overline{RC}_i} = 1/G_{\overline{RC}_i}$ Ω and $D_{\overline{RC}_i} = 1/C_{\overline{RC}_i}$ F^{-1}, respectively; transfer resistances of current-controlled voltage sources, $r_{mi}\Omega$. The set of impedance-based parameters describing the elements of the network is denoted by

$$\{z\} = \{R_i, L_i, D_i, \alpha_i, Z_{0i}, R_{\overline{RC}_i}, D_{\overline{RC}_i}, r_{mi}\}. \tag{1a}$$

On an admittance basis one considers the dual-controlled sources, which are voltage-controlled current sources with transfer admittance g_{mi}, and the set of elements corresponding to those in (1a) are

$$\{y\} = \{G_i, \Gamma_i, C_i, \gamma_i, Y_{0i}, G_{\overline{RC}_i}, C_{\overline{RC}_i}, g_{mi}\} \tag{1b}$$

Manuscript received April 10, 1970; revised October 6, 1970. This work was supported by the National Research Council of Canada through NRC Grant A5277. R. Seviora would like to acknowledge financial assistance under a Mary H. Beatty Fellowship at the University of Toronto. This paper was presented at the 13th Midwest Symposium on Circuit Theory, Minneapolis, Minn., May 1970.

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