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NOTE

A MATRIX OF COMBINATORIAL NUMBERS RELATED TO THE SYMMETRIC GROUPS

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A matrix $T=(t_{ik})$ is introduced, the coefficients of which are defined by $t_{ik}:=(i^k/(ik)!)\sum_{x\in S_n} a_i(x)^k$, $i, k\in\mathbf{N}=\{1,2,3,\ldots,\}$, where $a_i(x)$ denotes the number of i cycles in the element x of the symmetric group S_n. It is shown that these numbers are natural numbers, that they are easy to evaluate, and that they serve very well in order to formulate an infinite number of characterizations of multiply transitive subgroups of symmetric groups in terms of the cycle structure of their elements.

Let G denote a subgroup of the symmetric group S_n acting on the set $\boldsymbol{n}:=\{1,\ldots,n\}$. For $g\in G$ we denote by $a_i(g)$ the number of cyclic factors of g which are of length i once g is written as usual as a product of disjoint cyclic permutations.

For $i, k\in\mathbf{N}$ we put

$$\tau_{i,k}:=\frac{1}{(ik)!}\sum_{x\in S_{ik}} a_i(x)^k. \tag{1}$$

Let us first mention two results due to Foulkes [1] concerning these numbers.

If $0\leq b_i\in\mathbf{Z}$, $1\leq i\leq n$, are such that $\sum_i ib_i=n$, then

$$\frac{1}{n!}\sum_{x\in S_n} a_1(x)^{b_1}\cdots a_n(x)^{b_n}=\prod_{\substack{i=1\\ b_i>0}}^{n}\tau_{i,b_i}. \tag{2}$$

Furthermore the $\tau_{i,k}$ can be expressed in terms of the Stirling numbers $S(k,j)$ of the second kind as follows:

$$\tau_{i,k}=\sum_{j=1}^{k}\frac{S(k,j)}{i^j}. \tag{3}$$

These numbers $\tau_{i,k}$ turned out to be useful in order to characterize multiply transitive subgroups G of S_n. For by a result of Tsuzuku [4], $G\leq S_n$ is t-fold transitive, if and only if for any $0\leq b_i\in\mathbf{Z}$, $1\leq i\leq t$, the following holds:

$$\sum_{i=1}^{t} ib_i\leq t\Rightarrow\frac{1}{|G|}\sum_{g\in G} a_1(g)^{b_1}\cdots a_t(g)^{b_t}=\frac{1}{n!}\sum_{x\in S_n} a_1(x)^{b_1}\cdots a_t(x)^{b_t}. \tag{4}$$

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We now simply put

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$$t_{i,k} := i^k \tau_{i,k}, \tag{5}$$

for any $i, k \in \mathbf{N}$, and obtain from (3) that the following is true:

$$t_{i,k} = \frac{i^k}{(ik)!} \sum_{x \in S_n} a_i(x)^k = \sum_{j=1}^{k} i^{k-j} S(k, j) \in \mathbf{N}. \tag{6}$$

These combinatorial numbers form a matrix

$$T := (t_{i,k}).$$

Using (6) and one of the well-known explicit formulae for the Stirling numbers of the second kind, the first coefficients of T can be evaluated easily with the aid of a programmable pocket calculator, some of them are shown below:

k=1 k=2 k=3 k=4

$$\begin{array}{ll}
 & i=0 \\
 & i=1 \\
 & i=2 \\
 & i=3 \\
 & i=4
\end{array}
\qquad
T = \begin{bmatrix}
1 & 2 & 5 & 15 & 52 & 203 & 877 & 4140 \cdots \\
1 & 3 & 11 & 49 & 257 & 1539 & 10299 & 75905 \cdots \\
1 & 4 & 19 & 109 & 742 & 5815 & 51193 & 498118 \cdots \\
1 & 5 & 29 & 201 & 1657 & 15821 & 170389 & 2032785 \cdots \\
1 & 6 & 41 & 331 & 3176 & 35451 & 447981 & 6282416 \cdots \\
1 & 7 & 55 & 505 & 5497 & 69823 & 1007407 & 16157905 \cdots \\
1 & 8 & 71 & 729 & 8842 & 125399 & 2026249 & 36458010 \cdots \\
1 & 9 & 89 & 1009 & 13457 & 210105 & 3747753 & 74565473 \cdots \\
1 & 10 & 109 & 1351 & 19612 & 333451 & 6493069 & 141264820 \cdots \\
1 & 11 & 131 & 1761 & 27601 & 506651 & 10674211 & 251686881 \cdots \\
\vdots & & & & & & &
\end{bmatrix}
\qquad
\begin{array}{l}
\text{Bell \#'s} \\
4211 \\
4212 \\
4213 \\
5011 \\
5012
\end{array}
\tag{7}$$

& 111673 ↙

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The first row contains the sequence of Bell numbers since

$t_{1,k}$ →

$$t_{i,k} = \frac{1}{k!} \sum a_1(x)^k, \tag{8}$$

i.e. $t_{1,k}$ is by Burnside's lemma equal to the number of orbits of the action of S_k on the set of k-tuples $(i_1, \ldots, i_k)$, $i_j \in \boldsymbol{n}$, by

$$x(i_1, \ldots, i_k) := (x(i_1), \ldots, x(i_k)).$$

That this number is the Bell number follows immediately from (6) and

$$B_k = \sum_{j=1}^{k} S(k, j),$$

as well as from the fact that there is an obvious bijection from the set of orbits of S_k on the set of k-tuples onto the set of partitions of the set $\boldsymbol{k}$. For

$$(i_1, \ldots, i_k) \text{ in orbit of } (j_1, \ldots, j_k) \Leftrightarrow \forall 1 \leqslant \nu, \mu \leqslant k (i_\nu = i_\mu \Leftrightarrow j_\nu = j_\mu). \tag{9}$$

The sequences which form the other rows are not contained in Sloane's book [3].

By application of Tsuzuku's result we now obtain the following

Theorem. *$G \leq S_n$ is t-fold transitive, if and only if for any $0 \leq b_i \in \mathbf{Z}$, $1 \leq i \leq t$, the following holds:*

$$\sum_{1}^{t} i b_i \leq t \Rightarrow \frac{1}{|G|} \sum_{g \in G} a_1(g)^{b_1} \cdots a_t(g)^{b_t} = \prod_{\substack{i=1 \\ b_i > 0}}^{t} \frac{t_{i,b_i}}{i^{b_i}}.$$

Let us conclude with a typical example! If $G \leq S_n$ is 4-fold transitive, then we obtain that

$$\frac{1}{|G|} \sum_{g} a_1(g)^2 a_2(g) = \frac{t_{1,2}}{1^2} \cdot \frac{t_{2,1}}{2^1} = 2 \cdot \frac{1}{2} = 1,$$

and also

$$\frac{1}{|G|} \sum_{g} a_2(g)^2 = \frac{t_{2,2}}{2^2} = \frac{3}{4}$$

(cf. [2, V, 20.5]), so that we have now a very easy and direct method at hand to formulate all the analogous characterizations of multiple transitive groups in terms of the average cycle structure of their elements.

References

[1] H.O. Foulkes, Group transitivity and a multiplicative function of a partition. J. Combinatorial Theory 9 (1970) 261–266.

[2] B. Huppert, Endliche Gruppen I (Springer-Verlag, Berlin, 1967).

[3] N. Sloane, A Handbook of Integer Sequences (Academic Press, New York, 1973).

[4] T. [illegible]uki, On multiple transitivity of permutation groups, Nagoya Math. J. 18 (1961) 93–109.

Discrete Math. 21 #3 (Mar. 1978) 319–321.

A mat. of comb. #s related to the sym. gp.

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