OEIS A005045

Richard J. Mathar*
Leiden Observatory, P.O. Box 9513, 2300 RA Leiden, The Netherlands
(Dated: May 15, 2008)

The equivalence between the generating function and the sum formula in sequence A005045 is proven for the three subsequences $n \equiv 0 (\mathrm{mod} 12)$, $n \equiv 1 (\mathrm{mod} 12)$ and $n \equiv 2 (\mathrm{mod} 12)$. The remaining 9 cases of the congruence could be attacked along the same line of argument.

I. GENERATING FUNCTION CONJECTURE

The conjectured ordinary generating function (o.g.f.) of sequence A005045 of the Online Encyclopedia of Integer Sequences (OEIS) [1] is [2, 3]

$$A005045(x) = x^2 \frac{1+x-x^3-x^5+x^6}{(1+x^2)(1+x+x^2)(1+x)^2(1-x)^5} = x^2+3x^3+6x^4+10x^5+17x^6+\cdots \tag{1}$$

Its decomposition in partial fractions is

$$\begin{aligned} A005045(x) &= -\frac{257}{3456(1-x)} - \frac{11}{32(1-x)^2} + \frac{3}{64(1+x)^2} + \frac{4+2x}{27(1+x+x^2)} + \frac{35}{288(1-x)^3} + \frac{1-x}{16(1+x^2)} \\ &\quad - \frac{11}{128(1+x)} + \frac{1}{24(1-x)^5} + \frac{1}{12(1-x)^4}. \end{aligned} \tag{2}$$

Reverse lookup of each term in the OEIS puts this into the format

$$\begin{aligned} A005045(n) &= -\frac{257}{3456}A000012(n) - \frac{11}{32}A000027(n+1) + \frac{3}{64}A105811(n+3) + \frac{2}{27}A099837(n+3) \\ &\quad + \frac{35}{288}A000217(n+1) + \frac{1}{16}A057077(n+1) - \frac{11}{128}A033999(n) \\ &\quad + \frac{1}{24}A000332(n+4) + \frac{1}{12}A000292(n+1) \end{aligned} \tag{3}$$

$$\begin{aligned} &= -\frac{257}{3456} - \frac{11}{32}(n+1) + \frac{3}{64}(-1)^n(n+1) + \frac{2}{27}[\overline{2,-1,-1}] + \frac{35}{288}\binom{n+2}{2} + \frac{1}{16}[\overline{1,-1,-1,1}] \\ &\quad - \frac{11}{128}(-1)^n + \frac{1}{24}\binom{n+4}{4} + \frac{1}{12}\binom{n+3}{3} \end{aligned} \tag{4}$$

$$\begin{aligned} &= \frac{3}{64}(-1)^n(n+1) + \frac{2}{27}[\overline{2,-1,-1}] + \frac{1}{16}[\overline{1,-1,-1,1}] \\ &\quad - \frac{11}{128}(-1)^n - \frac{593}{3456} + \frac{5}{64}n + \frac{59}{288}n^2 + \frac{1}{32}n^3 + \frac{1}{576}n^4 \end{aligned} \tag{5}$$

where the notation $[\overline{a_0, a_1, a_2, \cdots}]$ represents a periodically extended sequence (see Appendix). Expansion of the denominator of (1),

$$(1+x^2)(1+x+x^2)(1+x)^2(1-x)^5 = 1-2x+x^3+2x^5-2x^6-x^8+2x^{10}-x^{11}, \tag{6}$$

extracts in addition the recurrence

$$a(n) = 2a(n-1) - a(n-3) - 2a(n-5) + 2a(n-6) + a(n-8) - 2a(n-10) + a(n-11) \tag{7}$$

as a corollary to the conjecture. The recurrence reaches 11 values backwards—correlated to the appearance of periods of length 3 and 4 (effective period length 12, the least common multiple) in (5). We make this more explicit by

---

*URL: http://www.strw.leidenuniv.nl/~mathar splitting (5) into the 12 polynomial representations as a function of n mod 12,

$$A005045(n) = \frac{59}{288}n^2 + \frac{1}{32}n^3 + \frac{1}{576}n^4 + \begin{cases} \frac{1}{8}n, & n = 12b, \\ -\frac{155}{576} + \frac{1}{32}n, & n = 12b+1, \\ -\frac{25}{72} + \frac{1}{8}n, & n = 12b+2, \\ \frac{5}{64} + \frac{1}{32}n, & n = 12b+3, \\ -\frac{2}{9} + \frac{1}{8}n, & n = 12b+4, \\ -\frac{155}{576} + \frac{1}{32}n, & n = 12b+5, \\ -\frac{1}{8} + \frac{1}{8}n, & n = 12b+6, \\ -\frac{83}{576} + \frac{1}{32}n, & n = 12b+7, \\ -\frac{2}{9} + \frac{1}{8}n, & n = 12b+8, \\ -\frac{3}{64} + \frac{1}{32}n, & n = 12b+9, \\ -\frac{25}{72} + \frac{1}{8}n, & n = 12b+10, \\ -\frac{83}{576} + \frac{1}{32}n, & n = 12b+11. \end{cases} \tag{8}$$

In some cases, the right hand side factorizes:

- $A005045(n) = n(n^3 + 18n^2 + 118n + 72)/576$ if $n \equiv 0 \bmod 12$
- $A005045(n) = (n-1)(n^3 + 19n^2 + 137n + 155)/576$ if $n \equiv 1 \bmod 12$
- $A005045(n) = (n+2)(n^3 + 16n^2 + 86n - 100)/576$ if $n \equiv 2 \bmod 12$
- $A005045(n) = (n-1)(n^3 + 19n^2 + 137n + 155)/576$ if $n \equiv 5 \bmod 12$
- $A005045(n) = (n+1)(n^3 + 17n^2 + 101n - 83)/576$ if $n \equiv 7 \bmod 12$
- $A005045(n) = (n+2)(n^3 + 16n^2 + 86n - 100)/576$ if $n \equiv 10 \bmod 12$
- $A005045(n) = (n+1)(n^3 + 17n^2 + 101n - 83)/576$ if $n \equiv 11 \bmod 12$

II. SUM FORMULA

The task is to show that (8) equals

$$S(n) = \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2 \rfloor - m} \begin{cases} n - 2i + m + 1, & m + r \neq i/2; \\ \lfloor \frac{n-2i+m}{2} \rfloor + 1, & m + r = i/2, \end{cases} \tag{9}$$

where

$$n \equiv \begin{cases} 3k, & n = 0 \bmod 3, \\ 3k-1, & n = 2 \bmod 3, \\ 3k-2, & n = 1 \bmod 3. \end{cases} \tag{10}$$

Further below we'll use the evaluation of the r-sum

$$S(n) = \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2 \rfloor} \begin{cases} (\frac{i-1}{2} - m + 1)(n - 2i + m + 1), & i \text{ odd}; \\ \lfloor \frac{n+m}{2} \rfloor - i + 1 + (\frac{i}{2} - m)(n - 2i + m + 1) & i \text{ even}. \end{cases} \tag{11}$$

In anticipation of a structure with 12 terms involved in the recurrence, we note the associated values of b and k:

$$n \equiv \begin{cases} 12b, & n = 0 \bmod 12,\ k = 4b \\ 12b+1, & n = 1 \bmod 12,\ k = 4b+1 \\ 12b+2, & n = 2 \bmod 12,\ k = 4b+1 \\ 12b+3, & n = 3 \bmod 12,\ k = 4b+1 \\ 12b+4, & n = 4 \bmod 12,\ k = 4b+2 \\ 12b+5, & n = 5 \bmod 12,\ k = 4b+2 \\ 12b+6, & n = 6 \bmod 12,\ k = 4b+2 \\ 12b+7, & n = 7 \bmod 12,\ k = 4b+3 \\ 12b+8, & n = 8 \bmod 12,\ k = 4b+3 \\ 12b+9, & n = 9 \bmod 12,\ k = 4b+3 \\ 12b+10, & n = 10 \bmod 12,\ k = 4b+4 \\ 12b+11, & n = 11 \bmod 12,\ k = 4b+4 \end{cases} \tag{12}$$

$$S(n) = \begin{cases} \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b, & n = 0 \bmod 12, & k = 4b \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+1, & n = 1 \bmod 12, & k = 4b+1 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+2, & n = 2 \bmod 12, & k = 4b+1 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+3, & n = 3 \bmod 12, & k = 4b+1 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+4, & n = 4 \bmod 12, & k = 4b+2 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+5, & n = 5 \bmod 12, & k = 4b+2 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+6, & n = 6 \bmod 12, & k = 4b+2 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+7, & n = 7 \bmod 12, & k = 4b+3 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+8, & n = 8 \bmod 12, & k = 4b+3 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+9, & n = 9 \bmod 12, & k = 4b+3 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+10, & n = 10 \bmod 12, & k = 4b+4 \\ \sum_{i=1}^{n-k} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+11, & n = 11 \bmod 12, & k = 4b+4 \end{cases} \tag{13}$$

$$= \begin{cases} \sum_{i=1}^{8b} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b, \\ \sum_{i=1}^{8b} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+1, \\ \sum_{i=1}^{8b+1} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+2, \\ \sum_{i=1}^{8b+2} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+3, \\ \sum_{i=1}^{8b+2} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+4, \\ \sum_{i=1}^{8b+3} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+5, \\ \sum_{i=1}^{8b+4} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+6, \\ \sum_{i=1}^{8b+4} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+7, \\ \sum_{i=1}^{8b+5} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+8, \\ \sum_{i=1}^{8b+6} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+9, \\ \sum_{i=1}^{8b+6} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+10, \\ \sum_{i=1}^{8b+7} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r), & n = 12b+11, \end{cases} \tag{14}$$

The simple strategy is to use this fan-out of the cases to track down each of the floor functions in the sums. For the first three, this is done explicitly; the other cases have not been worked out due to lack of enthusiasm:

1. Case $n = 12b$:

$$S(12b) = \sum_{i=1,3,5,\ldots}^{8b-1} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r) + \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-n)}^{\lfloor i/2\rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r),$$

$$= \sum_{i=1,3,5,\ldots}^{8b-1} \sum_{m=\max(0,2i-12b)}^{(i-1)/2} \sum_{r=0}^{(i-1)/2-m} c(i,m,r) + \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-12b)}^{i/2} \sum_{r=0}^{i/2-m} c(i,m,r),$$

$$= \sum_{i=1,3,5,\ldots}^{8b-1} \sum_{m=\max(0,2i-12b)}^{(i-1)/2} (\frac{i-1}{2} - m + 1)(12b - 2i + m + 1)$$

$$+ \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-12b)}^{i/2} [6b + \lfloor m/2\rfloor - i + 1 + (\frac{i}{2} - m)(12b - 2i + m + 1)]$$

$$= \sum_{i=0}^{4b-1} \sum_{m=\max(0,4i+2-12b)}^{i} (i-m+1)(12b-4i+m-1) + \sum_{i=0}^{4b-1} \sum_{m=\max(0,4i+4-12b)}^{i+1} [6b+\lfloor m/2\rfloor-2i-1+(i-m+1)(12b-4i+m-3)]$$

The sums over m are empty if $n = b = 0$, which shows that $S(n)$ equals (8) for that single case, and we deal only with $b \geq 1$ from here on.

$$S(12b) = \sum_{i=0}^{3b-1} \sum_{m=0}^{i} (i-m+1)(12b-4i+m-1) + \sum_{i=3b}^{4b-1} \sum_{m=4i+2-12b}^{i} (i-m+1)(12b-4i+m-1)$$
$$+ \sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} [6b + \lfloor m/2 \rfloor - 2i - 1 + (i-m+1)(12b-4i+m-3)]$$
$$+ \sum_{i=3b}^{4b-1} \sum_{m=4i+4-12b}^{i+1} [6b + \lfloor m/2 \rfloor - 2i - 1 + (i-m+1)(12b-4i+m-3)]$$

The intermediate sums with constant exponents are of the elementary form [4],

$$\sum_{x=0}^{s} x^k = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \binom{s+1}{j+1} j!, \tag{15}$$

and only the floor functions of $m/2$ need additional care,

$$S(12b) = \frac{3}{2}b + \frac{57}{2}b^2 + 50b^3 + 36b^4 + \sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} \lfloor m/2 \rfloor + \sum_{i=3b}^{4b-1} \sum_{m=4i+4-12b}^{i+1} \lfloor m/2 \rfloor. \tag{16}$$

$$\sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} \lfloor m/2 \rfloor + \sum_{i=3b}^{4b-1} \sum_{m=4i+4-12b}^{i+1} \lfloor m/2 \rfloor = \sum_{m=1}^{4b} \sum_{i=m-1}^{\lfloor m/4 \rfloor + 3b-1} \lfloor m/2 \rfloor$$
$$= \sum_{m=1,3,5,\ldots}^{4b-1} \sum_{i=m-1}^{\lfloor m/4 \rfloor + 3b-1} (m-1)/2 + \sum_{m=2,4,\ldots}^{4b} \sum_{i=m-1}^{\lfloor m/4 \rfloor + 3b-1} m/2$$
$$= \sum_{m=0}^{2b-1} \sum_{i=2m}^{\lfloor m/2+1/4 \rfloor + 3b-1} m + \sum_{m=0}^{2b-1} \sum_{i=2m+1}^{\lfloor m/2+1/2 \rfloor + 3b-1} (m+1)$$
$$= \sum_{m=0,2,4,\ldots}^{2b-2} \sum_{i=2m}^{m/2+3b-1} m + \sum_{m=1,3,\ldots}^{2b-1} \sum_{i=2m}^{(m-1)/2+3b-1} m + \sum_{m=0,2,\ldots}^{2b-2} \sum_{i=2m+1}^{m/2+3b-1} (m+1) + \sum_{m=1,3,\ldots}^{2b-1} \sum_{i=2m+1}^{(m+1)/2+3b-1} (m+1)$$
$$= \sum_{m=0}^{b-1} \sum_{i=4m}^{m+3b-1} 2m + \sum_{m=0}^{b-1} \sum_{i=4m+2}^{m+3b-1} (2m+1) + \sum_{m=0}^{b-1} \sum_{i=4m+1}^{m+3b-1} (2m+1) + \sum_{m=0}^{b-1} \sum_{i=4m+3}^{m+3b} (2m+2) = 4b^3 + b^2.$$

Inserted into (16) we have

$$S(12b) = \frac{3}{2}b + \frac{59}{2}b^2 + 54b^3 + 36b^4,$$

and insertion of $b = n/12$ yields the first line of (8). This concludes the proof for $n = 12b$.

2. Case $n = 12b + 1$:

$$S(12b+1) = \sum_{i=1,3,5,\ldots}^{8b-1} \sum_{m=\max(0,2i-n)}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2 \rfloor - m} c(i,m,r) + \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-n)}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2 \rfloor - m} c(i,m,r),$$

$$= \sum_{i=1,3,5,\ldots}^{8b-1} \sum_{m=\max(0,2i-12b-1)}^{(i-1)/2} \sum_{r=0}^{(i-1)/2-m} c(i,m,r) + \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-12b-1)}^{i/2} \sum_{r=0}^{i/2-m} c(i,m,r),$$

$$= \sum_{i=1,3,5,\ldots}^{8b-1} \sum_{m=\max(0,2i-12b-1)}^{(i-1)/2} (\frac{i-1}{2} - m + 1)(12b - 2i + m + 2)$$

$$+ \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-12b-1)}^{i/2} [6b + \lfloor (m+1)/2 \rfloor - i + 1 + (\frac{i}{2} - m)(12b - 2i + m + 2)]$$

$$= \sum_{i=0}^{4b-1} \sum_{m=\max(0,4i+1-12b)}^{i} (i-m+1)(12b-4i+m) + \sum_{i=0}^{4b-1} \sum_{m=\max(0,4i+3-12b)}^{i+1} [6b + \lfloor \frac{m+1}{2} \rfloor - 2i - 1 + (i-m+1)(12b-4i+m-2)]$$

The sums over i are empty if $b = 0, n = 1$, which shows that $S(n)$ equals (8) for that single case, and we deal only with $b \geq 1$ from here on.

$$S(12b+1) = \sum_{i=0}^{3b-1} \sum_{m=0}^{i} (i-m+1)(12b-4i+m) + \sum_{i=3b}^{4b-1} \sum_{m=4i+1-12b}^{i} (i-m+1)(12b-4i+m)$$

$$+ \sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} [6b + \lfloor \frac{m+1}{2} \rfloor - 2i - 1 + (i-m+1)(12b-4i+m-2)]$$

$$+ \sum_{i=3b}^{4b-1} \sum_{m=4i+3-12b}^{i+1} [6b + \lfloor \frac{m+1}{2} \rfloor - 2i - 1 + (i-m+1)(12b-4i+m-2)].$$

Again, only the floor functions of $(m+1)/2$ need additional care,

$$S(12b+1) = \frac{9}{2}b + \frac{79}{2}b^2 + 62b^3 + 36b^4 + \sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} \lfloor (m+1)/2 \rfloor + \sum_{i=3b}^{4b-1} \sum_{m=4i+3-12b}^{i+1} \lfloor (m+1)/2 \rfloor. \qquad (17)$$

$$\sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} \lfloor (m+1)/2 \rfloor + \sum_{i=3b}^{4b-1} \sum_{m=4i+3-12b}^{i+1} \lfloor (m+1)/2 \rfloor = \sum_{m=1}^{4b} \sum_{i=m-1}^{\lfloor (m-3)/4 \rfloor + 3b} \lfloor (m+1)/2 \rfloor$$

$$= \sum_{m=1,3,5,\ldots}^{4b-1} \sum_{i=m-1}^{\lfloor (m-3)/4 \rfloor + 3b} (m+1)/2 + \sum_{m=2,4,\ldots}^{4b} \sum_{i=m-1}^{\lfloor (m-3)/4 \rfloor + 3b} m/2$$

$$= \sum_{m=0}^{2b-1} \sum_{i=2m}^{\lfloor m/2 - 1/2 \rfloor + 3b} (m+1) + \sum_{m=0}^{2b-1} \sum_{i=2m+1}^{\lfloor m/2 - 1/4 \rfloor + 3b} (m+1)$$

$$= \sum_{m=0,2,4,\ldots}^{2b-2} \sum_{i=2m}^{m/2+3b-1} (m+1) + \sum_{m=1,3,\ldots}^{2b-1} \sum_{i=2m}^{(m-1)/2+3b} (m+1) + \sum_{m=0,2,\ldots}^{2b-2} \sum_{i=2m+1}^{m/2+3b-1} (m+1) + \sum_{m=1,3,\ldots}^{2b-1} \sum_{i=2m+1}^{(m-1)/2+3b} (m+1)$$

$$= \sum_{m=0}^{b-1} \sum_{i=4m}^{m+3b-1} (2m+1) + \sum_{m=0}^{b-1} \sum_{i=4m+2}^{m+3b} (2m+2) + \sum_{m=0}^{b-1} \sum_{i=4m+1}^{m+3b-1} (2m+1) + \sum_{m=0}^{b-1} \sum_{i=4m+3}^{m+3b} (2m+2) = 4b^3 + 5b^2 + 2b.$$

Inserted into (17) we have

$$S(12b+1) = \frac{13}{2}b + \frac{89}{2}b^2 + 66b^3 + 36b^4,$$

and insertion of $b = (n-1)/12$ yields the same expression as in the second line of (8). This concludes the proof for $n = 12b+1$.

3. Case $n = 12b+2$:

$$S(12b+2) = \sum_{i=1,3,5,\ldots}^{8b+1} \sum_{m=\max(0,2i-n)}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r) + \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-n)}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2\rfloor - m} c(i,m,r),$$

$$= \sum_{i=1,3,5,\ldots}^{8b+1} \sum_{m=\max(0,2i-12b-2)}^{(i-1)/2} \sum_{r=0}^{(i-1)/2-m} c(i,m,r) + \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-12b-2)}^{i/2} \sum_{r=0}^{i/2-m} c(i,m,r),$$

$$= \sum_{i=1,3,5,\ldots}^{8b+1} \sum_{m=\max(0,2i-12b-2)}^{(i-1)/2} (\frac{i-1}{2} - m + 1)(12b - 2i + m + 3)$$

$$+ \sum_{i=2,4,6,\ldots}^{8b} \sum_{m=\max(0,2i-12b-2)}^{i/2} [6b + \lfloor m/2 \rfloor - i + 2 + (\frac{i}{2} - m)(12b - 2i + m + 3)]$$

$$= \sum_{i=0}^{4b} \sum_{m=\max(0,4i-12b)}^{i} (i-m+1)(12b-4i+m+1) + \sum_{i=0}^{4b-1} \sum_{m=\max(0,4i+2-12b)}^{i+1} [6b+\lfloor m/2\rfloor - 2i + (i-m+1)(12b-4i+m-1)]$$

$$S(12b+2) = \sum_{i=0}^{3b-1} \sum_{m=0}^{i} (i-m+1)(12b-4i+m+1) + \sum_{i=3b}^{4b} \sum_{m=4i-12b}^{i} (i-m+1)(12b-4i+m+1)$$

$$+ \sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} [6b + \lfloor m/2 \rfloor - 2i + (i-m+1)(12b-4i+m-1)]$$

$$+ \sum_{i=3b}^{4b-1} \sum_{m=4i+2-12b}^{i+1} [6b + \lfloor m/2 \rfloor - 2i + (i-m+1)(12b-4i+m-1)]$$

Again, only the floor functions of $m/2$ need additional care,

$$S(12b+2) = 1 + \frac{33}{2}b + \frac{119}{2}b^2 + 74b^3 + 36b^4 + \sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} \lfloor m/2 \rfloor + \sum_{i=3b}^{4b-1} \sum_{m=4i+2-12b}^{i+1} \lfloor m/2 \rfloor. \tag{18}$$

$$\sum_{i=0}^{3b-1} \sum_{m=0}^{i+1} \lfloor m/2 \rfloor + \sum_{i=3b}^{4b-1} \sum_{m=4i+2-12b}^{i+1} \lfloor m/2 \rfloor = \sum_{m=1}^{4b} \sum_{i=m-1}^{\lfloor m/4-1/2 \rfloor + 3b} \lfloor m/2 \rfloor$$

$$= \sum_{m=1,3,5,\ldots}^{4b-1} \sum_{i=m-1}^{\lfloor m/4-1/2\rfloor+3b} (m-1)/2 + \sum_{m=2,4,\ldots}^{4b} \sum_{i=m-1}^{\lfloor m/4-1/2\rfloor+3b} m/2$$

$$= \sum_{m=0}^{2b-1} \sum_{i=2m}^{\lfloor m/2-1/4\rfloor+3b} m + \sum_{m=0}^{2b-1} \sum_{i=2m+1}^{\lfloor m/2\rfloor+3b} (m+1)$$

$$= \sum_{m=0,2,4,\ldots}^{2b-2} \sum_{i=2m}^{m/2+3b-1} m + \sum_{m=1,3,\ldots}^{2b-1} \sum_{i=2m}^{(m-1)/2+3b} m + \sum_{m=0,2,\ldots}^{2b-2} \sum_{i=2m+1}^{m/2+3b} (m+1) + \sum_{m=1,3,\ldots}^{2b-1} \sum_{i=2m+1}^{(m-1)/2+3b} (m+1)$$

$$= \sum_{m=0}^{b-1} \sum_{i=4m}^{m+3b-1} 2m + \sum_{m=0}^{b-1} \sum_{i=4m+2}^{m+3b} (2m+1) + \sum_{m=0}^{b-1} \sum_{i=4m+1}^{m+3b} (2m+1) + \sum_{m=0}^{b-1} \sum_{i=4m+3}^{m+3b} (2m+2) = 4b^3 + 3b^2.$$

Inserted into (18) we have

$$S(12b+2) = 1 + \frac{33}{2}b + \frac{125}{2}b^2 + 78b^3 + 36b^4,$$

and insertion of $b = (n-2)/12$ yields the first line of (8). This concludes the proof for $n = 12b+2$.

4. Case $n = 12b + 3$:
5. Case $n = 12b + 4$:
6. Case $n = 12b + 5$:
7. Case $n = 12b + 6$:
8. Case $n = 12b + 7$:
9. Case $n = 12b + 8$:
10. Case $n = 12b + 9$:
11. Case $n = 12b + 10$:
12. Case $n = 12b + 11$:

APPENDIX A: SOME PERIODIC SEQUENCES

Optionally, with o.g.f. $(2+x)/(1+x+x^2)$,

$$\overline{[2,-1,-1]} = \frac{2^n}{(\sqrt{3}i-1)^n} + \frac{(-2)^n}{(\sqrt{3}i+1)^n} = 2\left(-\frac{1}{2}\right)^n \sum_{l=0,1,\ldots}^{\lfloor n/2 \rfloor} \binom{n}{2l}(-3)^l \tag{A1}$$

can be inserted (i is the imaginary unit). Optionally, with o.g.f. $(1-x)/(1+x^2)$,

$$\overline{[1,-1,-1,1]} = \frac{1}{2}[(1+i)i^n + (1-i)(-i)^n] = \Re i^n - \Im i^n. \tag{A2}$$

These have not been used above.

[1] N. J. A. Sloane, Notices Am. Math. Soc. **50**, 912 (2003), http://www.research.att.com/~njas/sequences/, arXiv:math.CO/0312448, URL http://www.research.att.com/~njas/sequences/.

[2] S. Plouffe, Ph.D. thesis, Universite du Québec à Montréal (1992), URL http://www.lacim.uqam.ca/~plouffe/articles/MasterThesis.pdf.

[3] S. Plouffe, *1031 generating functions and conjectures* (1992), URL http://www.lacim.uqam.ca/~plouffe/articles/FonctionsGeneratrices.pdf.

[4] M. Z. Spivey, Discr. Math. **307**, 3130 (2007), URL http://math.ups.edu/~mspivey/BriefCV.html.