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E. J. [Morgan] Billington
& NJAS

Correspondence 1978 - 1991
17 numbered
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1 sequence

New Sequence

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DEPARTMENT OF MATHEMATICS
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6th April 1978

Name: 3 × 3 matrices with row and column sums n.

Ref: M05.

Dr. N.J.A. Sloane
Mathematics Research Centre
Bell Telephone Laboratories Inc.
Murray Hill
NEW JERSEY U.S.A.

Dear Dr. Sloane,

Our Mathematics Library has recently acquired two copies of your book, "A handbook of integer sequences". I am currently writing my Ph.D. thesis, and I noticed that a (combinatorial) sequence which arises from my work is not listed in your book. It begins as follows:

2 3 4 5 6 7 8 (struck through)

1, 3, 6, 10, 17, 25, 37, 51, . . . ,

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and it arises in the following manner.

Let P_μ denote a 3×3 matrix with entries chosen from $\{0,1,2,\ldots,\mu-1\}$, such that each row and each column of P_μ sums to μ, where μ is any positive integer greater than 1. No row or column of P_μ may have two zero entries, since μ is not an allowed entry of the matrix. For fixed μ, I call two such matrices P_μ equivalent if one can be obtained from the other by permuting rows, permuting columns, or taking the transpose. Let $n(\mu)$ be the number of non-equivalent matrices P_μ. Then the above sequence is the sequence

$$n(2), n(3), n(4), \ldots, n(9), \ldots.$$

I am including with this letter four pages copied from an appendix of my thesis, which include the non-equivalent P_μ for small μ, and a formula for $n(\mu)$, for arbitrary μ.

In your book you mention the existence of supplements; if any are available, I would be very grateful to receive one.

Yours sincerely

Elizabeth J. Morgan

Elizabeth J. Morgan

5045 (2)

$c_1(1)$: 1356, 1378, 1456, 1478, 1458, 1467 and complements, which is isomorphic to design VII via (67)(23), or

$c_1(2)$: 1358, 1367, 1456, 1456, 1478, 1478 and complements, which is mapped to design VIII by (142)(578).

Similarly case (c_2) may be completed in two ways:

$c_2(1)$: 1458, 1467, 1356, 1356, 1378, 1378 and complements, which is mapped to design VIII via (1432)(5678), or

$c_2(2)$: 1456, 1478, 1356, 1378, 1358, 1367 and complements, which is a design with 38 blocks of type A and 4 of type B, design X.

This completes the proof of Theorem 3.3.8. □

We shall conclude this section with some results on $(8, 14\mu, 7\mu, 4, 3\mu)$; $\lambda_3 = \mu$ 3-designs for arbitrary μ. As noted after (3.10), there are μ possible block types; a block of type $(i, 4(\mu-i), 6(\mu+i), 4(\mu-i), i-1)$ we shall call of type A_i, for $1 \le i \le \mu$. If we let α_i be the number of blocks of type A_i, we know that $\sum_{i=1}^{\mu} \alpha_i = 14\mu$, and (from Lemma 3.3.5), α_μ is 0, 2μ, 6μ or 14μ. Clearly there are other constraints on the values of the α_i; since $\binom{8}{4} = 70$, there are only $70 = 14 \times 5$ distinct blocks, and since $b = 14\mu$, if $\mu > 5$, we cannot have all blocks of type A_1. This clearly generalises:

LEMMA 3.3.9. *If* $\mu > 5w$ *then there exists at least* $2i$ *blocks of type* A_i *where* $i > w$.

Proof. If $\mu > 5w$ then $b = 14\mu > 70w$ and so even if each of the 70 distinct 4-sets is repeated w times, there are not enough blocks. So some block must occur i times where $i > w$; that is, some block must be of type A_i where $i > w$. Since both this block and its complement must each occur i times, there are at least $2i$ blocks

(3)

of type A_i with $i > w$. □

Now we shall consider in turn the four possible values of α_μ, and obtain all designs for the cases of the three non-zero values. If $\alpha_\mu = 14\mu$, there is only one design; it consists of μ identical copies of the unique $(8,14,7,4,3)$; $\lambda_3 = 1$ 3-design. The following lemma deals with the case $\alpha_\mu = 6\mu$.

LEMMA 3.3.10. *There are* $\left[\frac{\mu}{2}\right]$ *distinct* $(8,14\mu,7\mu,4,3\mu)$; $\lambda_3 = \mu$ *3-designs with* 6μ *blocks of type* A_μ.

Proof. Without loss of generality we may take the 6μ blocks of type A_μ to consist of μ copies each of:

1234	5678
1256	3478
1278	3456 .

There are 8μ more blocks, 2μ each of:

$$13{*}{*} \qquad 24{*}{*} \qquad 14{*}{*} \qquad 23{*}{*} \; . \tag{3.14}$$

The asterisks must be replaced by 2μ each of the pairs 57, 58, 67, 68. Also there must be μ each of the 3-sets 135, 136, 137, 138, 145, 146, 147, 148. Suppose that the block 1357 occurs m times. Since $a_\mu = a_0-1$ for all the possible block types, we know that the complement of each block must appear as a block. A simple check of pairs and 3-sets then shows that the blocks (3.14) must be as follows.

Block	Number of copies	Complement
1357	m	2468
1358	μ-m	2467
1368	m	2457
1367	μ-m	2458
1457	μ-m	2368
1458	m	2367
1468	μ-m	2357
1467	m	2358

(4)

Possible values for m are: $1, 2, \cdots, \left[\frac{\mu}{2}\right]$, where as usual $[n]$ denotes the largest integer less than or equal to n. The value 0 for m is ruled out because we have assumed that there are only 6μ blocks of type A_μ. Also we require $m \leq \left[\frac{\mu}{2}\right]$, for a design with $m > \left[\frac{\mu}{2}\right]$ is isomorphic to a design with $m < \left[\frac{\mu}{2}\right]$; this can be seen by interchanging 7 and 8. So there are $\left[\frac{\mu}{2}\right]$ distinct designs with $\alpha_\mu = 6\mu$. A typical one of these has $\alpha_\mu = 6\mu$, $\alpha_m = 8m$, $\alpha_{\mu-m} = 8(\mu-m)$ and other α_i, $i \neq \mu, m, \mu-m$, are zero. □

Now suppose that there are 2μ blocks of type A_μ; let them be μ copies each of 1234 and 5678. The remaining blocks must be 2μ each of

12** 13** 14**
34** 24** 23**

where asterisks are to be replaced by 2μ each of 56, 57, 58, 67, 68, 78. Once the blocks containing 1 are determined, the remaining blocks are also determined, since each block is complemented. Note that if, say, 56 occurs m times with 12, then 78 must also occur m times with 12; this follows because if m of the blocks 12** are 1256, then there are $\mu-m$ blocks 125* and $\mu-m$ blocks 126*, leaving m blocks 1278. We can make an array denoting the possible number of blocks of each type; the number n_2, for example, means that there are n_2 blocks 1357:

	56	57	58	67	68	78
12**	m_1	m_2	$\mu-m_1-m_2$	$\mu-m_1-m_2$	m_2	m_1
13**	n_1	n_2	$\mu-n_1-n_2$	$\mu-n_1-n_2$	n_2	n_1
14**	$\mu-m_1-n_1$	$\mu-m_2-n_2$	$m_1+m_2+n_1+n_2-\mu$	$m_1+m_2+n_1+n_2-\mu$	$\mu-m_2-n_2$	$\mu-m_1-n_1$

Let P_μ denote the 3×3 matrix

5

$$\begin{bmatrix} m_1 & m_2 & \mu-m_1-m_2 \\ n_1 & n_2 & \mu-n_1-n_2 \\ \mu-m_1-n_1 & \mu-m_2-n_2 & m_1+m_2+n_1+n_2-\mu \end{bmatrix} . \qquad (3.15)$$

We insist that $m_1, m_2, n_1, n_2 \in \{0,1,2,\cdots,\mu-1\}$, and that no row or column of P_μ contains two zero entries, or else we would have a design with more that 2μ blocks of type A_μ. Note also that permuting the rows or columns of P_μ or taking its transpose will lead to an isomorphic design; so the number of non-equivalent 3-designs with $\alpha_\mu = 2\mu$ is equal to the number of non-equivalent matrices P_μ, where two matrices are considered to be non-equivalent if one cannot be obtained from the other by permuting rows and columns or taking the transpose. For example, when $\mu = 2$ there is only one matrix, $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, and this corresponds to the (8,28,14,4,6); $\lambda_3 = 2$ 3-design with $\alpha_2 = 2\mu = 4$ and $\alpha_1 = 24$, design III of Theorem 3.3.7. When $\mu = 3$ there are three matrices, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$; these correspond respectively to designs VI, V and IV of Appendix E.

The following theorem is proved in Appendix F. It gives the number, $n(\mu)$, of non-equivalent matrices P_μ for arbitrary μ.

THEOREM F.

$$n(\mu) = \sum_{i=1}^{\mu-\ell} \sum_{m=\max\{0,2i-\mu\}}^{[i/2]} \sum_{r=0}^{[i/2]-m} c_{imr} ,$$

where $c_{imr} = \begin{cases} \mu-2i+m+1 & \textit{when } m+r \neq i/2 , \\ \left[\frac{\mu-2i+m+2}{2}\right] & \textit{when } m+r = i/2 , \end{cases}$

and ℓ *is given by:* $\mu = 3\ell$, $3\ell-1$ *or* $3\ell-2$, *according as* $\mu \equiv 0$, 2 *or* 1 *(modulo 3).*

(This excludes the case $\mu = 2$, *but trivially* $n(2) = 1$.*)* □

6

Appendix F also lists the possible numbers of blocks of type A_i, $1 \leq i \leq \mu-1$, when $\alpha_\mu = 2\mu$, for $2 \leq \mu \leq 6$. We now have the following lemma.

LEMMA 3.3.11. *There are* $n(\mu)$ *distinct* $(8,14\mu,7\mu,4,3\mu)$; $\lambda_3 = \mu$ *3-designs with* 2μ *blocks of type* A_μ. □

In the case $\alpha_\mu = 0$, with no blocks of type A_μ, it seems somewhat harder to find all the 3-designs. Also it is not even clear whether each design will be uniquely determined by the number of blocks of each type.

If there is one block of type A_r, then there are (at least) $2r$ blocks of type A_r, and the design must have the "skeleton" form as follows:

				Number of blocks
r each	1234	5678		2r
	123*	567*	[μ-r each of	
μ-r each	124*	568*	1,2,3,4,5,6,	8μ-8r
	134*	578*	7,8 to replace	
	234*	678*	asterisks]	
	12**	34**	[μ+r each of 56,	
μ+r each	13**	24**	57,58,67,68,78 to	6μ+6r
	14**	23**	replace asterisks]	
				14μ blocks

We may assume that there are the following numbers of blocks:

Block	Number of them	Block	Number of them
1235	a_1	1345	c_1
1236	a_2	1346	c_2
1237	a_3	1347	c_3
1238	a_4	1348	c_4
1245	b_1	2345	d_1
1246	b_2	2346	d_2
1247	b_3	2347	d_3
1248	b_4	2348	d_4

Here $\sum_{i=1}^{4} a_i = \sum_{i=1}^{4} b_i = \sum_{i=1}^{4} c_i = \sum_{i=1}^{4} d_i = \mu - r.$

We also take blocks as described by the following array:

	56	57	58	67	68	78
12	m_1	m_2	m_3	m_4	m_5	m_6
13	n_1	n_2	n_3	n_4	n_5	n_6
14	p_1	p_2	p_3	p_4	p_5	p_6

Here

$$m_6 = r - \mu + m_1 + a_1 + b_1 + a_2 + b_2,$$
$$n_6 = r - \mu + n_1 + a_1 + c_1 + a_2 + c_2,$$
$$p_6 = r - \mu + p_1 + b_1 + c_1 + b_2 + c_2, \quad \text{(from rows)}$$

and $\sum_{i=1}^{6} m_i = \sum_{i=1}^{6} n_i = \sum_{i=1}^{6} p_i = \mu + r.$

Also

$$m_1 + n_1 + p_1 + m_6 + n_6 + p_6 = \mu + r,$$
$$m_2 + n_2 + p_2 + m_5 + n_5 + p_5 = \mu + r,$$
$$m_3 + n_3 + p_3 + m_4 + n_4 + p_4 = \mu + r. \quad \text{(From columns.)}$$

There are then several constraints, obtained by counting 3-sets $\{x,y,z\}$ where at least one of x, y, z belongs to $\{1,2,3,4\}$ and at least one to $\{5,6,7,8\}$. For example, a count of the 3-set $\{1,2,5\}$ yields $a_1 + b_1 + m_1 + m_2 + m_3 = \mu$.

A general result for arbitrary r and μ seems too involved, and will not be attempted here.

AS045 (8)

APPENDIX F

Numbers of non-equivalent matrices P_μ

(See section 3.3)

μ	*No. of matrices*, $n(\mu)$
2	1
3	3
4	6
5	10
6	17
7	25
8	37
9	51

Numbers of blocks of each type:

$\mu = 2$

A_1	A_2
24	4

$\mu = 3$

A_1	A_2	A_3
36	0	6
12	24	6
20	16	6

$\mu = 4$

A_1	A_2	A_3	A_4
0	48	0	8
24	24	0	8
12	0	36	8
16	32	0	8
16	8	24	8
12	24	12	8

$\mu = 5$

A_1	A_2	A_3	A_4	A_5
12	0	0	48	10
12	24	8	12	10
8	24	12	16	10
16	0	12	32	10
0	24	36	0	10
4	32	24	0	10
24	0	36	0	10
16	32	12	0	10
12	24	24	0	10
12	48	0	0	10

$\mu = 6$

A_1	A_2	A_3	A_4	A_5	A_6
12	0	0	0	60	12
12	8	0	32	20	12
8	8	36	0	20	12
8	16	12	16	20	12
16	0	0	16	40	12
0	24	0	48	0	12
4	16	36	16	0	12
0	40	0	32	0	12
8	24	24	16	0	12
12	16	12	32	0	12
16	16	24	16	0	12
24	0	0	48	0	12
0	0	72	0	0	12
8	16	48	0	0	12
12	24	36	0	0	12
8	40	24	0	0	12
0	72	0	0	0	12

(9)

$\mu = 2.$ $\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$

$\mu = 3.$ $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ $\begin{pmatrix}2&1&0\\1&0&2\\0&2&1\end{pmatrix}$ $\begin{pmatrix}2&0&1\\1&1&1\\0&2&1\end{pmatrix}$

$\mu = 4.$ $\begin{pmatrix}2&2&0\\2&0&2\\0&2&2\end{pmatrix}$ $\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$ $\begin{pmatrix}3&1&0\\1&0&3\\0&3&1\end{pmatrix}$ $\begin{pmatrix}2&1&1\\2&1&1\\0&2&2\end{pmatrix}$ $\begin{pmatrix}3&1&0\\1&2&1\\0&1&3\end{pmatrix}$ $\begin{pmatrix}3&1&0\\1&1&2\\0&2&2\end{pmatrix}$

$\mu = 5.$ $\begin{pmatrix}4&1&0\\1&0&4\\0&4&1\end{pmatrix}$ $\begin{pmatrix}4&1&0\\1&1&3\\0&3&2\end{pmatrix}$ $\begin{pmatrix}4&1&0\\1&2&2\\0&2&3\end{pmatrix}$ $\begin{pmatrix}4&1&0\\1&3&1\\0&1&4\end{pmatrix}$ $\begin{pmatrix}3&2&0\\2&0&3\\0&3&2\end{pmatrix}$ $\begin{pmatrix}3&2&0\\2&1&2\\0&2&3\end{pmatrix}$

$\begin{pmatrix}3&1&1\\1&3&1\\1&1&3\end{pmatrix}$ $\begin{pmatrix}3&1&1\\1&2&2\\1&2&2\end{pmatrix}$ $\begin{pmatrix}3&1&1\\2&2&1\\0&2&3\end{pmatrix}$ $\begin{pmatrix}2&2&1\\2&1&2\\1&2&2\end{pmatrix}$

$\mu = 6.$ $\begin{pmatrix}5&1&0\\1&0&5\\0&5&1\end{pmatrix}$ $\begin{pmatrix}5&1&0\\1&1&4\\0&4&2\end{pmatrix}$ $\begin{pmatrix}5&1&0\\1&2&3\\0&3&3\end{pmatrix}$ $\begin{pmatrix}5&1&0\\1&3&2\\0&2&4\end{pmatrix}$ $\begin{pmatrix}5&1&0\\1&4&1\\0&1&5\end{pmatrix}$

$\begin{pmatrix}4&2&0\\2&0&4\\0&4&2\end{pmatrix}$ $\begin{pmatrix}4&2&0\\2&1&3\\0&3&3\end{pmatrix}$ $\begin{pmatrix}4&2&0\\2&2&2\\0&2&4\end{pmatrix}$ $\begin{pmatrix}4&1&1\\2&2&2\\0&3&3\end{pmatrix}$ $\begin{pmatrix}4&1&1\\2&3&1\\0&2&4\end{pmatrix}$

$\begin{pmatrix}4&1&1\\1&3&2\\1&2&3\end{pmatrix}$ $\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}$

$\begin{pmatrix}3&3&0\\3&0&3\\0&3&3\end{pmatrix}$ $\begin{pmatrix}3&2&1\\3&1&2\\0&3&3\end{pmatrix}$ $\begin{pmatrix}3&2&1\\2&1&3\\1&3&2\end{pmatrix}$ $\begin{pmatrix}3&2&1\\2&2&2\\1&2&3\end{pmatrix}$ $\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}$

Let $\mu = 3\ell$, $3\ell-1$ or $3\ell-2$, according as $\mu \equiv 0$, 2 or 1 (modulo 3). (This excludes the case $\mu = 2$, but trivially $n(2) = 1$.)

A5045

THEOREM F.

$$n(\mu) = \sum_{i=1}^{\mu-\ell} \sum_{m=\max\{0,2i-\mu\}}^{[i/2]} \sum_{r=0}^{[i/2]-m} c_{imr},$$

where

$$c_{imr} = \begin{cases} \mu - 2i + m + 1 & \text{when } m+r \neq \frac{i}{2}, \\ \left[\dfrac{\mu - 2i + m + 2}{2}\right] & \text{when } m+r = \frac{i}{2}. \end{cases}$$

Proof. The counting of possible 3×3 matrices depends on the following ordering; it ensures that only non-equivalent matrices are counted.

Let $P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where every row and column sums to μ, and rows and columns are arranged so that, in order,

(i) $a_{11} \geq a_{ij}$ for $1 \leq i,j \leq 3$;

(ii) $a_{12} + a_{21}$ is then to be as large as possible;

(iii) $a_{21} \geq a_{12}$;

(iv) if $a_{12} = a_{13}$, then $a_{22} \geq a_{23}$.

The entry a_{11} may equal $\mu-1, \mu-2, \cdots, \ell$; in other words, $a_{11} = \mu-i$ for $i = 1,2,\cdots,\mu-\ell$. So the first column may be $\begin{pmatrix} \mu-i & \cdot & \circ \\ i-m & \cdot & \cdot \\ m & \cdot & \cdot \end{pmatrix}$ where $i-m$ and m are both less than or equal to $\mu-i$, and $i-m \geq m$. So $2m \leq i$, or $m \leq [i/2]$, and also $i-m \leq \mu-i$, or $m \geq 2i-\mu$.

Now the first row may be taken as follows:

$$\begin{pmatrix} \mu-i & i-m-r & m+r \\ i-m & \cdot & \circ \\ m & \cdot & \cdot \end{pmatrix} \tag{F.1}$$

where $i-m-r \geq m+r$, or $r \leq [i/2]-m$, and also $i-m-r \leq \mu-i$, or $r \geq (2i-\mu)-m$; since $m \leq 2i-\mu$, this merely gives $r \geq 0$.

It remains to find the number of choices available for the entry a_{22} in (F.1); once this entry is determined, the whole matrix is known. So we must verify that c_{imr} is as given in the statement of the theorem. We claim that a_{22} may take all integer values satisfying

$$a_{13} \leq a_{22} \leq \mu - (a_{12} + a_{21}) , \tag{F.2}$$

(11)

that is, $\mu - (a_{12} + a_{21} + a_{13}) + 1$ values (with an additional condition if $a_{12} = a_{13}$). For if $a_{22} < a_{13}$, then $a_{22} < m+r$, so that $a_{32} = \mu - a_{12} - a_{22} > \mu - (i-m-r) - (m+r) = \mu - i = a_{11}$; but this contradicts condition (i). Also if $a_{22} > \mu - (a_{12}+a_{21})$, then $a_{22}+a_{21} > \mu-a_{12}$, or (since each row sum equals μ), $\mu-a_{23} > \mu-a_{12}$, or $a_{12} > a_{23}$. However $a_{33}+a_{23}+a_{13} = a_{11}+a_{12}+a_{13}$, so that $a_{12} > a_{23}$ implies $a_{33} > a_{11}$, again contradicting condition (i). If $a_{12} = a_{13}$, to ensure that condition (iv) holds we must let a_{22} take values greater than or equal to a_{23}. So instead of taking $\mu - (a_{12}+a_{21}+a_{13}) + 1$ different values or $a_{11}-a_{21}+1$ values or $\mu-2i+m+1$ values, a_{22} can only take $\left\lfloor \frac{\mu-2i+m+2}{2} \right\rfloor$ values. Finally, note that $a_{12} = a_{13}$ implies that $i = 2(m+r)$. □

As an example, consider $\mu = 8 = 3 \times 3 - 1$, so $\ell = 3$.

$$n(8) = \sum_{i=1}^{5} \sum_{m=\max\{0,2i-8\}}^{[i/2]} \sum_{r=0}^{[i/2]-m} c_{imr} =$$

$$c_{100} + (c_{200} + c_{201} + c_{210}) + (c_{300} + c_{301} + c_{310}) +$$

$$(c_{400} + c_{401} + c_{402} + c_{410} + c_{411} + c_{420}) + c_{520} .$$

$c_{100} = 7$: $\begin{pmatrix} 7 & 1 & 0 \\ 1 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 0,1,2,3,4,5,6$.

$c_{200} = 5$: $\begin{pmatrix} 6 & 2 & 0 \\ 2 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 0,1,2,3,4$.

$c_{201} = 3$: $\begin{pmatrix} 6 & 1 & 1 \\ 2 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 3,4,5$.

$c_{210} = 3$: $\begin{pmatrix} 6 & 1 & 1 \\ 1 & & \\ 1 & & \end{pmatrix}$ where $a_{22} = 4,5,6$.

$c_{300} = 3$: $\begin{pmatrix} 5 & 3 & 0 \\ 3 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 0,1,2$.

$c_{301} = 3$: $\begin{pmatrix} 5 & 2 & 1 \\ 3 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 1,2,3$.

$c_{310} = 4$: $\begin{pmatrix} 5 & 2 & 1 \\ 2 & & \\ 1 & & \end{pmatrix}$ where $a_{22} = 1,2,3,4$.

$c_{400} = 1$: $\begin{pmatrix} 4 & 4 & 0 \\ 4 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 0$.

$c_{401} = 1$: $\begin{pmatrix} 4 & 3 & 1 \\ 4 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 1$.

$c_{402} = 1$: $\begin{pmatrix} 4 & 2 & 2 \\ 4 & & \\ 0 & & \end{pmatrix}$ where $a_{22} = 2$.

$c_{410} = 2$: $\begin{pmatrix} 4 & 3 & 1 \\ 3 & & \\ 1 & & \end{pmatrix}$ where $a_{22} = 1,2$.

$c_{411} = 1$: $\begin{pmatrix} 4 & 2 & 2 \\ 3 & & \\ 1 & & \end{pmatrix}$ where $a_{22} = 3$.

$c_{420} = 2$: $\begin{pmatrix} 4 & 2 & 2 \\ 2 & & \\ 2 & & \end{pmatrix}$ where $a_{22} = 3,4$.

$c_{520} = 1$: $\begin{pmatrix} 3 & 3 & 2 \\ 3 & & \\ 2 & & \end{pmatrix}$ where $a_{22} = 2$.

Hence $n(8) = 7+5+3+3+3+3+4+1+1+1+2+1+2+1 = 37$.

5045

(13)

Department of Mathematics
University of Queensland
ST. LUCIA, BRISBANE, 4067

(Your ref: MH-1216-NJAS-lay)

28 Oct 1980.

Dear Dr. Sloane,

The only reference to my sequence {1, 3, 6, 10, ...} giving the number of 3×3 matrices with constant row and column sum is in the Amer. Math. Soc. Notices (Abstract, for the Jan '79 Biloxi Meeting) Jan '79 Issue 191, vol 26 no 1, page A 27. I've not published it anywhere. Otherwise the reference to my thesis is:

Elizabeth J. Morgan, Construction of block designs and related results, University of Queensland, (1978), Ph. D. thesis.

Sorry for the delay in replying to your letter but I've not been well for a few weeks.

Yours sincerely,

Elizabeth J. Billington (née Morgan)

f91

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AT&T (14)

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May 28, 1991

Professor Elizabeth J. Morgan
c/o Professor Ann Penfold Street
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St. Lucia QD 4067
AUSTRALIA

Dear Professor Morgan:

Thirteen years ago you sent me the sequence 1, 3, 6, 10, 17,... from your thesis! Can you please supply me with an exact reference?

Best regards,

N. J. A. Sloane

f91

5045

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The University of Queensland
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(15)

12th June 1991.

Professor N.J.A. Sloane,
AT&T Bell Laboratories,
600 Mountain Avenue,
Murray Hill,
NJ 07974–2070,
U.S.A.

Dear Professor Sloane,

Thank you for your May 28th letter, via Anne Street. The sequence 1, 3, 6, 10, 17, ... that I sent you many years ago has not really appeared in print. I gave a ten minute talk at the 85th Annual Meeting of the American Mathematical Society in Biloxi in January 1979. The abstract appeared in
Notices of the AMS, Volume 26, Number 1 (January 1979, Issue 191), page A-27 (abstract number 763–05–13).
I didn't publish the result more widely because I seem to remember that from questions afterwards (possibly from Frank Harary? But I hardly knew him then!) similar work had been done by graph theorists; since this was only a sideline of my main work on designs, I pursued the matter no further. The reference to more detail in my thesis is:
Elizabeth J. Morgan, "Construction of block designs and related results", The University of Queensland, 1978.
But of course this is only available from the library here at the University of Queensland.

Sorry I don't have a better reference for you! I've enclosed a photostat of the relevant pages of my thesis, but that's not really what you want.

Yours sincerely,

Elizabeth Billington

Elizabeth J. Billington

PS. I changed my name in 1980 when I married, even for publications: I now publish under the name Billington.

Notices
of the American Mathematical Society

January 1979, Issue 191
Volume 26, Number 1, Pages 1—82, A-1—A-190
Providence, Rhode Island USA
ISSN 0002-9920

763-05-13 ELIZABETH J. MORGAN, University of Queensland, St. Lucia, Queensland 4067, Australia. <u>On 3 × 3 integer matrices with constant row and column sum.</u> Preliminary report.

Let $P = [p_{ij}]$ be a 3×3 matrix with $p_{ij} \in \{0,1,2,\ldots,\mu-1\}$, such that $\sum_{i=1}^{3} p_{ik} = \sum_{j=1}^{3} p_{kj} = \mu$, for $k = 1,2,3$. Any matrix which can be obtained from P by permuting rows and columns of P or by taking the transpose of P is said to be equivalent to P. The number, $n(\mu)$, of such inequivalent matrices for any $\mu \geq 2$, is given by $n(\mu) = \sum_{i=1}^{\mu-\ell} \sum_{m=\max\{0,2i-\mu\}}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2 \rfloor - m} c_{imr}$, where $c_{imr} = \mu-2i+m+1$ if $m+r \neq \frac{1}{2}i$, and $c_{imr} = \lfloor \frac{1}{2}(\mu-2i+m+2) \rfloor$ if $m+r = \frac{1}{2}i$, and where $\mu = 3\ell$, $3\ell-2$ or $3\ell-1$ according as $\mu = 0, 1$, or 2 (modulo 3). The problem of determining $n(\mu)$ arose in connection with the enumeration of all non-isomorphic doubly balanced designs on eight elements with block size four. (Received October 20, 1978.)
(Author introduced by Dr. Anne Penfold Street).

*763-05-14 WILLIAM B. POUCHER, Abilene Christian University, Abilene, Texas 79601. <u>A note on intersection preserving embeddings of partial (n,4)-PBDs.</u> Preliminary report.

An (n,K)-PBD is a pairwise balanced design of order n and block sizes from K. If $|K| = 1$, then we replace K with its single element. It has been shown by Lindner and Rosa for Steiner Triple Systems and by the author for all other (n,K)-PBDs where $K \neq \{4\}$ that any collection of partial (n,K)-PBDs can be embedded in a collection of (v,K)-PBDs preserving initial blockset intersection. This paper solves the remaining problem for $K = \{4\}$. In addition it is shown THEOREM: Any partial SQS can be embedded in a $3 \longrightarrow 2$ resolvable partial SQS. (Received October 20, 1978.)

763-05-15 JACOB E. GOODMAN, City College (CUNY), New York, N.Y. 10031. <u>Configurations in the plane: isotopy and combinatorial equivalence.</u> Preliminary report.

A configuration $(P_1,\ldots,P_n)$ in R^2 is <u>nondegenerate</u> if no three points are collinear and no lines joining pairs of points are parallel. Let C, C' be nondegenerate n-configurations in R^2. C is <u>isotopic</u> to C' if there is a continuous family of nondegenerate configurations joining C and C'. Each pair P_i, P_j in C determines a line L_{ij} through 0, parallel to $\overline{P_iP_j}$; C is <u>combinatorially equivalent</u> to C' if the cyclic ordering of this family of lines agrees in both configurations. A <u>half-space</u> of C is the subset lying on one side of some line. A basic problem in combinatorial geometry is that of finding necessary and sufficient conditions for a family of subsets of an abstract n-set to be the family of all half-spaces in some nondegenerate embedding of C in R^2 (or in R^d, for that matter). We discuss the relation between combinatorial equivalence and isotopy, and state a conjectured solution of this problem involving sequences of permutations of the points of a configuration. (Received October 23, 1978.)

763-05-16 Jean H. Bevis, Georgia State University, Atlanta, Georgia 30303. <u>Determinants for variable adjacency matrices.</u> Preliminary report.

For a graph G=(V,E) the free idempotent commutative semiring F over E (considered as a set of lables or indeterminants) provides a natural setting for the study of path algebras over G. Although there are no additive inverses in F, a meaningful determinant function may be defined for matrices over F. This is obtained by using the symmetric difference of canonical representations of elements of F. Many of the results for determinants over the commutative ring of polynomials over E may be obtained in this new setting. In particular, versions of Harary's theorem for the variable adjacency matrix, and Kirchhoff's theorem for the variable incidence matrix are obtained for matrices over F. (Received October 23, 1978.)

763-05-17 HAI-PING KO, Oakland University, Rochester, Michigan 48063 and DIJEN K. RAY-CHAUDHURI, The Ohio State University, Columbus, Ohio 43210. <u>A multiplier theorem of group ring.</u>

Let G be an abelian group of exponent v^*. Write elements of the group ring Z[G] in the form $\sum_{g \in G} a_g x^g$, $a_g \in Z$. For every subset D of G, define $D(x) = \sum_{d \in D} x^d$. An integer t is called a multiplier of a member, d(x), of Z[G] if $(t, v^*) = 1$ and $d(x^t) = x^g d(x)$ for some $g \in G$. <u>Theorem.</u> Suppose $d(x)d(x^\alpha) = a + bH(x) + cG(x)$ in Z[G] for some integers a, b, c, a subgroup H of G, and a homomorphism $\alpha : G \to G$