CONWAY'S COSMOLOGICAL THEOREM

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1. Introduction

In [C], Conway introduced an operator on strings (finite sequences) of positive integers, the audioactive (or "look and say") operator. Usually, the integers involved are single digit numbers, and a string consisting of four ones and a single three, for instance, is written as 11113; in general, the terms of the sequence are referred to as digits regardless of their size. To apply the operator to the above string, describe it in words as "four ones, one three" and then translate this to the digit string 4113. More formally, suppose a string S is written as $d_1^{e_1} d_2^{e_2} \cdots d_n^{e_n}$, where d^e represents e consecutive digits d. Whenever this is done, it is understood that the grouping is maximal; that is, $d_i \neq d_{i+1}$ for $1 \leq i < n$. Then the result of applying the operator to S is the daughter sequence $\alpha(S) = e_1 d_1 e_2 d_2 \cdots e_n d_n$. Applying the operator repeatedly produces the descendants of S, $\alpha^n(S)$ for $n = 0, 1, \ldots$. A string that is equal to $\alpha^n(S)$ for some string S and integer n is called an *n-day-old* string.

Some strings of the form $S = LR$ (with L and R non-empty) have the property that the descendants of L and R do not interfere with one another, in the sense that $\alpha^n(S) = \alpha^n(L)\alpha^n(R)$ for all n. In this case, we say that S *splits*, and write $S = L.R$. A string that does not split is called an *atom*, or *element*, and every string splits uniquely into atoms, and is also called a *compound*. Conway proved that there is a finite set of *common* elements that is stable, in the sense that if S is a common element, all the elements appearing in $\alpha(S)$ are common, and that has the remarkable property that any sufficiently old descendant of an arbitrary string (other than $\emptyset$ and 22) involves all the common elements. Astoundingly, there are precisely 92 of these elements, and (less astoundingly) Conway gave them the names Hydrogen, Helium, ..., Uranium. He also found two infinite families of elements that are persistent, in the sense that each such element S appears in $\alpha^n(S)$ for some $n > 0$ (in fact for $n = 2$), and so keeps reappearing. These he called the isotopes of Neptunium and Plutonium, collectively known as the *transuranic* elements. His *Cosmological Theorem* asserts that there is some integer N such that any N-day-old string splits into common and transuranic elements. One proof of this was found by Conway and Richard Parker, and another by Mike Guy that showed that one could take $N = 24$. Both proofs were lost. Subsequently, Ekhad and Zeilberger [EZ] gave a proof consisting of a Maple program written by Zeilberger and executed by Ekhad[1], together with a short explanation of what the program does and why its output constitutes a proof. This, however, only shows that one may take $N = 29$.

This paper discusses two proofs, both computer assisted, that indeed one may take $N = 24$. The C source code for the programs involved and some related programs, together with documentation, can be found at

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[1]Zeilberger is a human being and Ekhad is his computer.

`http://www.math.lsu/~lither/jhc/audioactive.tar.gz.`

The first proof is a minor adaptation of Ekhad and Zeilberger's. Almost all the work is done by the program `proof1`; the algorithm used, and why its execution constitutes a proof, are explained in the accompanying documentation, and this proof will not be discussed further here. The second proof is similar in spirit (I think) to Conway and Parker's. That consisted, according to Conway, of "a very subtle and complicated argument, which (almost) reduced the problem to tracking a few hundred cases", which were then "handled on dozens of sheets of paper (now lost)." The rest of this paper gives a complicated argument reducing the problem to tracking a finite number of cases. It is clearly not subtle enough, because the number is 3360. These cases are then checked by the program `proof2`.

That the number 24 cannot be improved was also shown by Guy, who found atoms of longevity 24, given on page 186 of [C]. Dropping the first two digits gives shorter elements of longevity 24, the isotopes of Thuselum, as illustrated in the documentation for my program `evolve`.

I assume familiarity with Conway's paper; in particular, all the named theorems referred to are found there. Our notation differs in the use of $\alpha(S)$ rather than S_1 for the daughter of S; subscripts on strings are just indices here. As in [C], a *chunk* of a string is a substring of consecutive digits. By a *run* of a string we mean a maximal chunk of identical digits, and by a *block* we mean a chunk made up of runs. A $\{1,3\}$-chunk or $\{1,3\}$-block is one containing only the digits 1 and 3. The *longevity* of a string S [EZ] is the minimum n for which $\alpha^n(S)$ is composed of common and transuranic elements.

2. Four-day-old strings

Recall that a string is parsed by inserting commas to indicate its derivation from a parent string. Conway gave restrictions on the strings or parsed strings that can be chunks of 1- and 2-day-old strings; we give further restrictions for 3- and 4-day-old strings.

Lemma 2.1.

(1) *A 1-day-old string has no chunks of the form* $x\mathbf{,}yx\mathbf{,}$.
(2) *A 2-day-old string has no chunks of the form* $\mathbf{,}xy\mathbf{,}$ *with* $x > 3$*, or* $3x3$.
(3) *A 3-day old string has no chunks of the form* $\mathbf{,}33\mathbf{,}$ *or* 31113.
(4) *A 4-day-old string has no chunks of the form* 233*,* 13112^1 *or* 13312^1.

Proof. Parts (1) and (2) are Conway's One-Day and Two-Day Theorems. In part (3), the first string must arise from a block 333, contradicting part (2). By part (1), the second can only be parsed as $3\mathbf{,}11\mathbf{,}13\mathbf{,}$, which arises from a chunk 313, again contrary to (2).

In part (4), the first string is parsed as $\mathbf{,}23\mathbf{,}3$ by (3), and so comes from a chunk $33xxx$. One parsing contradicts (1), the other (3). By (1) and (3), the other two must be parsed as $\mathbf{,}13\mathbf{,}11\mathbf{,}2x\mathbf{,}$ and $\mathbf{,}13\mathbf{,}31\mathbf{,}2x\mathbf{,}$ for some $x \neq 1$ or 2. These arise from $31xx$ and $3111xx$. By (2), $x = 3$, and we get a contradiction to (2) or (3). $\square$

We say that a string *looks 4 days old* if it has a parsing satisfying the conditions of Lemma 2.1. A chunk of a 4-day-old string need not look 4 days old. However, if S looks 4 days old, so does any block of S, as well as $\alpha(S)$.

Lemma 2.2. *The only maximal* $\{1, 3\}$*-chunks of a 4-day-old string are*

$$\emptyset, 1, 3, 11, 13, 31, 111, 113, 131, 133, 311,$$
$$1113, 1131, 1133, 1311, 1331, 3111, 3113,$$
$$11131, 11133, 11311, 11331, 13111, 13113, 13311, 31131, 31133,$$
$$111311, 111331, 113111, 113311, 311311, 311331,$$
$$1113111, 1113311, 3113111 \textit{ and } 3113311.$$

Proof. Let S be any $\{1, 3\}$-chunk of a 4-day-old string. Suppose length$(S) > 2$, and let S' be the maximal chunk of S that starts and ends with commas in the parsing of S. Then S' is non-empty and describes a $\{1, 3\}$-block P of a 3-day-old string that contains no runs of length 2. We claim that P contains at most one 3. For two consecutive 3s would be a run, and non-consecutive 3s would give a chunk 313 or 31113, both of which are impossible in 3-day-old strings. Hence P is one of the following eleven strings:

$$P_1 = 1, P_2 = 111, P_3 = 3, P_4 = 13, P_5 = 1113, P_6 = 31, P_7 = 3111,$$
$$P_8 = 131, P_9 = 11131, P_{10} = 13111 \text{ or } P_{11} = 1113111.$$

Therefore S is equal to $\alpha(P_i)$ for some i, or is obtained from it by prefixing or appending a single 1 or 3, or both. Only one of 1 or 3 may be prefixed (the one different from the first digit of P_i), giving a maximum of six possible S for each i. Also, P_4, P_5 and $P_8, \dots, P_{11}$ cannot follow a 1 or 3, since it would have to be a 3 and would give a chunk 313 or 31113. Further, P_2 and $P_7, \dots, P_{11}$ cannot be followed by a run of length three, since the resulting chunk would not parse. Finally, P_6 cannot be both preceded by a 1 and followed by a run of three. This means there are at most 39 such S, and one checks that all of them, plus the 7 strings of length at most 2, are listed above apart from 33, 331 and 3311 . The three exceptions cannot be maximal $\{1, 3\}$-chunks, because they cannot appear at the start of a 4-day old string by part (3) of Lemma 2.1, nor after a 2 by part (4), nor after $x > 3$ by parts (2) and (3). $\square$

Lemma 2.3. *Let S be an atom appearing in a 4-day-old string. A maximal non-empty* $\{1, 3\}$*-block of S not at the start of S is one of the following strings:*

$$1, 3, 11, 113, 133, 1131, 1133, 1331, 3111, 11311, 11331, 13311, 113111, \textit{ or } 113311.$$

Proof. Such a block B appears in a chunk $2B]$ or $2Bx$ ($x \neq 1, 3$) of S, and the non-empty strings listed in Lemma 2.2 but not here would cause a splitting $2.B]$ or $2.Bx$. $\square$

Let $\mathcal{S}^1_{\{1,3\}}$ be the set of strings in Lemma 2.2 and $\mathcal{S}^2_{\{1,3\}}$ the set of those in Lemma 2.3. Then any atom appearing in a 4-day-old string can be written in the form

(2.1) $M = CA_nB_{n-1}A_{n-1} \cdots B_1A_1$,

where:

(2.2) $C \in \mathcal{S}^1_{\{1,3\}}$;

(2.3) $B_i \in \mathcal{S}^2_{\{1,3\}}$ for $1 \le i < n$;

(2.4) $A_i = 2$, 22 or 222 for $1 < i \le n$; and

(2.5) $A_1 = \emptyset$, 2, 22, 222 or x for some $x > 3$.

(Here $n = 1$ and $M = CA_1$ is allowed.) We shall show that any such atom has longevity at most 20. It will be more convenient to work with strings satisfying

(2.6) $A_i = 2$, 22 or 222 for $1 \le i \le n$

in place of (2.4) and (2.5). We call a string *mature* if it looks 4 days old and has the form (2.1) subject to (2.2), (2.3) and (2.6).

Lemma 2.4. *If every mature string has longevity at most 20, then so does every atom appearing in a 4-day-old string.*

Proof. Let M be such an atom, written as in (2.1) subject to (2.2)–(2.5). Then M is mature unless $A_1 = \emptyset$ or $A_1 = x > 3$. Suppose that $A_1 = \emptyset$, and that M appears in $\alpha^4(S)$. If M is not the last atom in this string, the next atom is 22. If it is, $\alpha^4(S22)$ ends $M22$, so in either case $M22$ is a block of a 4-day old string, and is therefore mature. Since it splits as $M\boldsymbol{.}22$, it has the same longevity as M.

Finally, suppose $A_1 = x$ with $x > 3$. Since any occurence of x in a 2-day old string is neither preceded nor followed by x, in a 3-day-old string it is preceded by 1, and in a 4-day-old string by 11. Hence M ends $11x$ and it follows from the proof of the Ending Theorem that the last atom of $\alpha^k(M)$ is either Np or Pu for $k \ge 18$. Let M' be the string obtained from M by replacing x by 2. Now M does not end $1311x$, since the parsings **,**13**,**11**,**x and 1**,**31**,**1x**,** are both impossible. Hence M' is mature. Now $\alpha^k(M')$ is obtained from $\alpha^k(M)$ by replacing the final x by 2, and since $\alpha^{20}(M')$ is composed of common atoms, all atoms of $\alpha^{20}(M)$ apart from the last are common. □

The substrings in our decomposition of a mature string do not, of course, evolve independently. To keep track of their interactions with their neighbors, we regard each one as a state of a machine, as explained in the next section.

3. Machines and pipelines

For our purposes, a *machine* $\mathbb{M}$ consists of a set $\mathcal{S}$ of states, an input alphabet $\mathcal{I}$, an output alphabet $\mathcal{O}$ (all non-empty), and a program $\pi\colon \mathcal{S} \times \mathcal{I} \to \mathcal{O} \times \mathcal{S}$. The output function $\omega\colon \mathcal{S} \times \mathcal{I} \to \mathcal{O}$ and the transition function $\tau\colon \mathcal{S} \times \mathcal{I} \to \mathcal{S}$ are defined by composing π with the appropriate projections. If s is a string on $\mathcal{I}$ and $S \in \mathcal{S}$, we write $S \xrightarrow[t]{s} T$ to indicate that $\mathbb{M}$, receiving input s in state S, outputs the string t and ends in state T. (When s is a single letter, so is t, and this means that $\pi(S, s) = (t, T)$. The machine in question will be clear from the context.) In the machines we actually use, the output $\omega(S, x)$ is independent of x, so we sometimes write $S \xrightarrow[t]{s}$ to mean that $\mathbb{M}$, receiving input s in state S, outputs t (one letter longer than s), and ends in an unknown state. We also use the notation $S \xrightarrow[t]{s}$ for infinite sequences s and t on $\mathcal{I}$ and $\mathcal{O}$, which we call input and output *streams*. Finally, parts of s that do not affect the outcome will be suppressed.

Given machines $\mathbb{M}_1 = (\mathcal{S}_1, \mathcal{I}_1, \mathcal{O}_1, \pi_1)$ and $\mathbb{M}_2 = (\mathcal{S}_2, \mathcal{I}_2, \mathcal{O}_2, \pi_2)$ with $\mathcal{O}_1 = \mathcal{I}_2$, we can form the machine $\mathbb{M} = \mathbb{M}_2\mathbb{M}_1$, called a *pipeline*, by connecting the output of $\mathbb{M}_1$ to the input of $\mathbb{M}_2$. We have $\mathbb{M} = (\mathcal{S}, \mathcal{I}, \mathcal{O}, \pi)$, where $\mathcal{S} = \mathcal{S}_2 \times \mathcal{S}_1$, $\mathcal{I} = \mathcal{I}_1$, and $\mathcal{O} = \mathcal{O}_2$. The program for $\mathbb{M}$ should be intuitively clear; formally we have

$$\pi(S_2, S_1, x) = (\pi_2(S_2, \omega_1(S_1, x)), \tau_1(S_1, x)).$$

Clearly, we have a category whose arrows are machines; the order in which we write a pipeline corresponds to the "functions on the left" convention, and is also the natural one for our application of the idea. We now define three specific machines. The states, inputs, and outputs will all be digit strings. In fact, the inputs and outputs will be either the empty string $\emptyset$ or a single digit, so we can write the IO-streams without commas. By a *simple* string we mean a string on $\{1, 2, 3\}$ in which all runs have length at most 3.

The machine $\mathbb{A}$. Here $\mathcal{S} = \{2, 22, 222\}$, $\mathcal{I} = \{\emptyset, 2\}$ and $\mathcal{O} = \{\emptyset, 1, 3\}$. The program is given by

State 1	Input	Output	State 2
2^y, $y \neq 2$	x	y	$2x$
22	x	$\emptyset$	$22x$

The machine $\mathbb{B}$. Here $\mathcal{S}$ is the set of all non-empty simple strings that neither start nor end with a 2, $\mathcal{I} = \{\emptyset, 1, 3\}$ and $\mathcal{O} = \{\emptyset, 2\}$. For $S \in \mathcal{S}$, either $\alpha(S) \in \mathcal{S}$ or $\alpha(S) = 2T$ with $T \in \mathcal{S}$. The program is given by

State 1	Input	Output	State 2
S, $\alpha(S) \in \mathcal{S}$	x	$\emptyset$	$\alpha(S)x$
S, $\alpha(S) = 2T$	x	2	Tx

The machine $\mathbb{C}$. Here $\mathcal{S}$ is the set of all (possibly empty) simple strings that do not end with a 2, $\mathcal{I} = \{\emptyset, 1, 3\}$ and $\mathcal{O} = \{\emptyset\}$. The program is given by

State 1	Input	State 2
S	x	$\alpha(S)x$

Consider the machine $\mathbb{M} = \mathbb{C}\mathbb{A}_n\mathbb{B}_{n-1}\mathbb{A}_{n-1}\cdots\mathbb{B}_1\mathbb{A}_1$ for some n, where each $\mathbb{A}_i$ or $\mathbb{B}_i$ is a copy of $\mathbb{A}$ or $\mathbb{B}$. If $M = CA_nB_{n-1}A_{n-1}\cdots B_1A_1$ is a mature string, then $\mathbb{M}$ has $(C, A_n, B_{n-1}, A_{n-1}, \ldots, B_1, A_1)$ as a state, and if it is started from this state with input stream $\emptyset^\infty$, the product of the strings in the state after k inputs have been consumed will be $\alpha^k(M)$. We say that $\mathbb{M}$ is *run properly* if the initial state corresponds to a mature string and the input is $\emptyset^\infty$. We consider these machines in the next section.

Now suppose $\mathbb{M} = \mathbb{M}_n\cdots\mathbb{M}_1$ for some $n \geq 2$, where each $\mathbb{M}_i$ is a copy of $\mathbb{A}$, $\mathbb{B}$ or $\mathbb{C}$. Let $\mathbb{M} = (\mathcal{S}, \mathcal{I}, \mathcal{O}, \pi)$. ($\mathcal{I}$ is one of $\{\emptyset, 2\}$ and $\{\emptyset, 1, 3\}$, and $\mathcal{O}$ is one of $\{\emptyset\}$, $\{\emptyset, 2\}$ and $\{\emptyset, 1, 3\}$). We define a machine $\widehat{\mathbb{M}} = (\widehat{\mathcal{S}}, \mathcal{I}, \mathcal{O}, \widehat{\pi})$ with the same inputs and outputs as $\mathbb{M}$ as follows. $\widehat{\mathcal{S}}$ is the set of all non-empty simple strings whose first digit is not in $\mathcal{O}$ and whose last digit is in $\mathcal{I}$. For $S \in \widehat{\mathcal{S}}$, either $\alpha(S) \in \widehat{\mathcal{S}}$ or $\alpha(S) = yT$ with $y \in \mathcal{O}$ and $T \in \widehat{\mathcal{S}}$. The program is given by

State 1	Input	Output	State 2
S, $\alpha(S) \in \widehat{\mathcal{S}}$	x	$\emptyset$	$\alpha(S)x$
S, $\alpha(S) = yT$, $y \in \mathcal{O}$	x	y	Tx

Every state $(S_n, \ldots, S_1)$ of $\mathbb{M}$ determines a state $S_n \cdots S_1$ of $\widehat{\mathbb{M}}$. If $\mathbb{M}$ and $\widehat{\mathbb{M}}$ are started from corresponding states and given the same input, they will produce the same output, and at each stage their states will correspond. We call $\widehat{\mathbb{M}}$ the *flattening* of $\mathbb{M}$.

We now prove two lemmas about the machines $\mathbb{A}$ and $\mathbb{B}$. In following the evolution of these machines from specific states, we will always know an initial chunk of the state, and usually the whole state, so we do not use Conway's [and] notation. Our convention is that, for instance, 121 denotes a complete state, while $121\rangle$ denotes a state starting 121. We continue the convention that explicit exponents are maximal, so that if a state is written $1^1x^1\rangle$, it is implied that $x \neq 1$ and the state does not start $1xx$.

Lemma 3.1. *The output of $\mathbb{A}$ (for any initial state and input) does not contain $\emptyset 1$, 13 or 33.*

Proof. The possibilities for two consecutive steps are:

$$2 \xrightarrow[1]{\emptyset} 2 \xrightarrow[1]{}; \qquad 22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{}; \qquad 222 \xrightarrow[3]{\emptyset} 2 \xrightarrow[1]{};$$
$$2 \xrightarrow[1]{2} 22 \xrightarrow[\emptyset]{}; \qquad 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{}; \qquad 222 \xrightarrow[3]{2} 22 \xrightarrow[\emptyset]{}. \qquad \square$$

Lemma 3.2. *Let $L = \emptyset$ or 112, and let R be a string of the form $1^1x^1\rangle$, $1^3\rangle$ or $3^1x^{1 \text{ or } 2}\rangle$. Set $\bar{\omega}(\emptyset) = \emptyset$ and $\bar{\omega}(112) = 2$. Suppose that at some stage $\mathbb{B}$ has state LR. Then the subsequent output is $\bar{\omega}(L)^\infty$, and at all later stages $\mathbb{B}$ has state LR', where R' has one of the same three forms as R.*

Proof. Clearly the first output is $\bar{\omega}(L)$ and the next state is LR' where $R' = \alpha(R)$, $\alpha(R)1$ or $\alpha(R)3$. If $R = 1^1x^1\rangle$, then $\alpha(R) = 1^3\rangle$, and R' has the same form. If $R = 1^3\rangle$, then $\alpha(R) = 31^{1 \text{ or } 2}\rangle$, and R' could have a different form only if $\alpha(R)$ were equal to 311, which is impossible. Finally, if $R = 3^1x^{1 \text{ or } 2}\rangle$ then both $\alpha(R)$ and R' have the form $13(1 \text{ or } 2)\rangle$. The result follows. $\square$

When a state S satisfying the hypotheses of this lemma arises, we indicate this by writing S as $L\boldsymbol{.}R$.

4. Properly-run machines

Throughout this section, $\mathbb{M} = \mathbb{C}\mathbb{A}_n\mathbb{B}_{n-1}\mathbb{A}_{n-1}\cdots\mathbb{B}_1\mathbb{A}_1$ and we assume that $\mathbb{M}$ has an initial state corresponding to a mature string and input $\emptyset^\infty$. For $1 \le i \le n$, we let s_i be the input stream of $\mathbb{A}_i$, and t_i its output stream. Then s_i is a stream on $\{\emptyset, 2\}$, t_i is a stream on $\{\emptyset, 1, 3\}$, and $s_1 = \emptyset^\infty$.

Lemma 4.1.

1. *If $\emptyset 2$ appears in s_i, it is followed by 22, $\emptyset 2$, or $\emptyset^\infty$.*
2. *If $\emptyset\emptyset 2$ appears in s_i, it is followed by 222.*
3. *If $\emptyset^3$ appears in s_i, it is followed by $\emptyset^\infty$.*

Proof. If $i = 1$, this is trivially true, so suppose $i > 1$. Then s_i is also the output of the flattening $\mathbb{N}_i$ of $B_{i-1}A_{i-1}\cdots B_1A_1$ with input $\emptyset^\infty$. The state transitions below are for the machine $\mathbb{N}_i$.

(1) From the proof of the Starting Theorem, the state S of $\mathbb{N}_i$ just before $\emptyset 2$ appears must be $13^2\rangle$, $12^3\rangle$ or $12^2\rangle$. (It cannot be just 1.) Suppose first $S = 13^2\rangle$. Because S looks 4 days old, it must have the form $13^2x^{1 \text{ or } 2}\rangle$. Hence we have

$$13^2\rangle \xrightarrow[\emptyset]{} 1123(1 \text{ or } 2)\rangle \xrightarrow[2]{} 11213\rangle \xrightarrow[2]{} 11211\rangle \xrightarrow[2]{} 112\rangle \xrightarrow[2]{} 1\rangle,$$

so in this case $\emptyset 2$ is followed by 222. Now suppose $S = 12^3\rangle$. Then we have

$$12^3\rangle \xrightarrow[\emptyset]{} 1132^{1 \text{ or } 2}\rangle \xrightarrow[2]{} 113(1 \text{ or } 2)\rangle \xrightarrow[2]{} 113\rangle \xrightarrow[2]{} 1\rangle,$$

so in this case $\emptyset 2$ is followed by 22. Finally, suppose that $S = 12^2\rangle$. One possibility is

$$12^2\rangle \xrightarrow[\emptyset]{} 112^3\rangle \xrightarrow[2]{} 132\rangle,$$

and in this case $\emptyset 2$ is followed by $\emptyset^\infty$, by the Starting Theorem. Otherwise we have

$$12^2\rangle \underset{\emptyset}{\rightarrow} 112^2\rangle \underset{2}{\rightarrow} 122\rangle \underset{\emptyset}{\rightarrow} 11(\neq 1)\rangle \underset{2}{\rightarrow} 1\rangle$$

and in this case $\emptyset 2$ is followed by $\emptyset 2$.

(2) From the proof of the Starting Theorem, the state of $\mathbb{N}_i$ just before $\emptyset\emptyset 2$ appears must be $3^1x^3\rangle$, and the next step is $3^1x^3\rangle \underset{\emptyset}{\rightarrow} 13^2\rangle$. That this is followed by the output $\emptyset 2^4$ is part of the proof of part (1).

(3) From the proof of the Starting Theorem, the state of $\mathbb{N}_i$ just before $\emptyset^3$ appears must be $1^1x^1\rangle$, $1^3\rangle$ or $3^1x^{\neq 3}\rangle$, and the result follows. □

Lemma 4.2.

(1) *If* $\emptyset\emptyset 3$ *appears in* t_i*, it is followed by* $\emptyset 3$*,* $1\emptyset$*, or* 1^∞*.*
(2) *If* $\emptyset\emptyset\emptyset 3$ *appears in* t_i*, it is followed by* $\emptyset 3\emptyset$*.*
(3) *If* $\emptyset^4$ *appears in* t_i*, it is followed by* $\emptyset^\infty$*.*
(4) *If* $11\emptyset$ *appears in* t_i*, it is followed by* $3\emptyset$*,* $\emptyset 3$*, or* $\emptyset^\infty$*.*
(5) *If* $111\emptyset$ *appears in* t_i*, it is followed by* $3\emptyset 3$*.*
(6) *If* 1^4 *appears in* t_i*, it is followed by* 1^∞*.*

Proof. In cases (1)–(3) (resp. (4)–(6)), the state of $\mathbb{A}_i$ just before the given sequence appears in its output stream must be 22 (resp. 2), and the part of the input stream s_i about to be read must start $\emptyset 2$, $\emptyset\emptyset 2$ or $\emptyset^3$. In each case the possibilities for the continuation of s_i are given by the previous lemma, and we just need to compute the corresponding outputs. The following computations show the output starting with the given sequence.

(1)
$$\begin{gathered}
22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{2} 22 \xrightarrow[\emptyset]{2} 222 \underset{3}{\rightarrow};\\
22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{\emptyset} 2 \xrightarrow[1]{2} 22 \underset{\emptyset}{\rightarrow};\\
22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{\emptyset} 2 \xrightarrow[1^\infty]{\emptyset^\infty}.
\end{gathered}$$

(2)
$$22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{2} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{2} 22 \underset{\emptyset}{\rightarrow}.$$

(4)
$$\begin{gathered}
2 \xrightarrow[1]{\emptyset} 2 \xrightarrow[1]{2} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{2} 22 \underset{\emptyset}{\rightarrow};\\
2 \xrightarrow[1]{\emptyset} 2 \xrightarrow[1]{2} 22 \xrightarrow[\emptyset]{\emptyset} 22 \xrightarrow[\emptyset]{2} 222 \underset{3}{\rightarrow};\\
2 \xrightarrow[1]{\emptyset} 2 \xrightarrow[1]{2} 22 \xrightarrow[\emptyset^\infty]{\emptyset^\infty}.
\end{gathered}$$

(5)
$$2 \xrightarrow[1]{\emptyset} 2 \xrightarrow[1]{\emptyset} 2 \xrightarrow[1]{2} 22 \xrightarrow[\emptyset]{2} 222 \xrightarrow[3]{2} 22 \xrightarrow[\emptyset]{2} 222 \underset{3}{\rightarrow}.$$

(3,6)
$$22 \xrightarrow[\emptyset^\infty]{\emptyset^\infty}; \qquad 2 \xrightarrow[1^\infty]{\emptyset^\infty}.$$
□

Recall that the initial state of $\mathbb{A}_i$ (resp. $\mathbb{B}_i$) is the string A_i (resp. B_i) from the decomposition (2.1) of our mature string M.

Lemma 4.3. *If B_i is 133, 1131 or 3111, then t_i does not start with 3. If B_i is 1331 or 11311, t_i does not start with 1.*

Proof. We need to show that in the first three cases, A_i is not 222, and in the others it is not 2. This is so because M looks 4 days old. $\square$

In Lemmas 4.4–4.10, we determine the possible output streams s_{i+1} of $\mathbb{B}_i$ for the allowable initial states $B_i \in \mathcal{S}^2_{\{1,3\}}$. In each case, the proof consists of listing the evolutions of the state for all possible inputs until it becomes periodic. That we have considered all possible inputs t_i to $\mathbb{B}_i$ can be seen from Lemmas 4.3, 3.1 and 4.2. In most cases, that the output becomes periodic follows from Lemma 3.2, or from a case already dealt with. We shall therefore not give these justifications explicitly.

Lemma 4.4. *If B_i is one of the eight strings in the left column below, the output stream s_{i+1} of $\mathbb{B}_i$ is as indicated.*

B_i	s_{i+1}
1133, 11331, 113311	$2\emptyset^\infty$
1331, 13311	$\emptyset 2^\infty$
3111	$\emptyset^2 2^\infty$
11311	$2^5\emptyset^\infty$
113111	$2^2\emptyset^\infty$

Proof. We have

$$113^2\rangle \xrightarrow[2]{} .123\rangle \xrightarrow[\emptyset^\infty]{} ;$$

$$1331 \xrightarrow[\emptyset]{\emptyset \text{ or } 3} 112.31^2\rangle \xrightarrow[2^\infty]{} ;$$

$$13311 \xrightarrow[\emptyset]{} 112.321\rangle \xrightarrow[2^\infty]{} ;$$

$$3111 \xrightarrow[\emptyset]{\emptyset \text{ or } 1} 1331^{1 \text{ or } 2} \xrightarrow[\emptyset 2^\infty]{} ;$$

$$11311 \xrightarrow[2]{\emptyset \text{ or } 3} 11321^1\rangle \xrightarrow[2]{} 113121\rangle \xrightarrow[2]{} 1131112\rangle \xrightarrow[2]{} 11331\rangle \xrightarrow[2]{} .123\rangle \xrightarrow[\emptyset^\infty]{} ;$$

$$113111 \xrightarrow[2]{} 113^2\rangle \xrightarrow[2\emptyset^\infty]{} .$$

$\square$

Let s be a stream on $\{\emptyset, 2\}$ and t a finite or infinite sequence on $\{\emptyset, 1, 3\}$. In Lemmas 4.5–4.10, the notation $t \rightsquigarrow s$ means that, for the given value of B_i, if t_i starts with t, then $s_{i+1} = s$.

Lemma 4.5. *If $B_i = 1131$, we have $\emptyset \rightsquigarrow 2^6\emptyset^\infty$ and $1 \rightsquigarrow 2^3\emptyset^\infty$.*

Proof. We have

$$1131 \xrightarrow[2]{\emptyset} 11311 \xrightarrow[2^5\emptyset^\infty]{} \quad \text{and} \quad 1131 \xrightarrow[2]{1} 113111 \xrightarrow[2^2\emptyset^\infty]{} .$$

$\square$

Lemma 4.6. *If $B_i = 133$, we have 1 or $\emptyset\emptyset \rightsquigarrow \emptyset 2^\infty$, and $\emptyset 3 \rightsquigarrow \emptyset 2^4 \emptyset 2^3 \emptyset^\infty$.*

Proof. The first two cases are given by

$$133 \xrightarrow[\emptyset]{1} 112.31 \xrightarrow[2^\infty]{} \quad \text{and} \quad 133 \xrightarrow[\emptyset]{\emptyset} 1123 \xrightarrow[2]{\emptyset} 112.13 \xrightarrow[2^\infty]{},$$

and the third by

$$\begin{aligned} &133 \xrightarrow[\emptyset]{\emptyset} 1123 \xrightarrow[2]{3} 112133 \xrightarrow[2]{\emptyset \text{ or } 1} 1121123^1\rangle \xrightarrow[2]{} 112211213\rangle \\ &\xrightarrow[2]{} 122211211\rangle \xrightarrow[\emptyset]{} 11322112\rangle \xrightarrow[2]{} 1132221\rangle \xrightarrow[2]{} 11332\rangle \xrightarrow[2]{} .123\rangle \xrightarrow[\emptyset^\infty]{}. \end{aligned}$$

□

Lemma 4.7. *If $B_i = 3$, we have $\emptyset$ or $1 \rightsquigarrow \emptyset^\infty$; 31 or $3\emptyset\emptyset \rightsquigarrow \emptyset^2 2^\infty$; and $3\emptyset 3 \rightsquigarrow \emptyset^2 2^4 \emptyset 2^3 \emptyset^\infty$.*

Proof. We have

$$\begin{aligned} &3 \xrightarrow[\emptyset]{\emptyset} .13 \xrightarrow[\emptyset^\infty]{}; \quad 3 \xrightarrow[\emptyset]{1} .131 \xrightarrow[\emptyset^\infty]{}; \\ &3 \xrightarrow[\emptyset]{3} 133 \xrightarrow[\emptyset 2^\infty]{1}; \quad 3 \xrightarrow[\emptyset]{3} 133 \xrightarrow[\emptyset 2^\infty]{\emptyset\emptyset}; \\ &3 \xrightarrow[\emptyset]{3} 133 \xrightarrow[\emptyset 2^4 \emptyset 2^3 \emptyset^\infty]{\emptyset 3}. \end{aligned}$$

□

Lemma 4.8. *If $B_i = 113$, we have $1\emptyset \rightsquigarrow 2^7\emptyset^\infty$, $11 \rightsquigarrow 2^4\emptyset^\infty$, $\emptyset^a 3 \rightsquigarrow 2^{a+2}\emptyset^\infty$ for $0 \le a \le 3$, and $\emptyset^\infty \rightsquigarrow 2^\infty$.*

Proof. First,

$$113 \xrightarrow[2]{1} 1131 \xrightarrow[2^6\emptyset^\infty]{\emptyset} \quad \text{and} \quad 113 \xrightarrow[2]{1} 1131 \xrightarrow[2^3\emptyset^\infty]{1}.$$

Second, noting that $113 \xrightarrow[2]{\emptyset} 113$,

$$113 \xrightarrow[2^a]{\emptyset^a} 113 \xrightarrow[2]{3} 1133 \xrightarrow[2\emptyset^\infty]{} \quad \text{and} \quad 113 \xrightarrow[2^\infty]{\emptyset^\infty}.$$

□

Lemma 4.9. *If $B_i = 1$, we have*

$$\begin{aligned} &1 \rightsquigarrow \emptyset^\infty, \\ &31\emptyset \rightsquigarrow \emptyset 2^7 \emptyset^\infty, \quad 311 \rightsquigarrow \emptyset 2^4 \emptyset^\infty, \quad 3\emptyset^\infty \rightsquigarrow \emptyset 2^\infty, \\ &3\emptyset^a 3 \rightsquigarrow \emptyset 2^{a+2} \emptyset^\infty \quad \textit{for } 1 \le a \le 3, \\ &\emptyset 3 \rightsquigarrow \emptyset 2 \emptyset^\infty, \\ &\emptyset^2 31\emptyset \rightsquigarrow \emptyset 2 \emptyset 2^7 \emptyset^\infty, \quad \emptyset^2 311 \rightsquigarrow \emptyset 2 \emptyset 2^4 \emptyset^\infty, \quad \emptyset^2 3\emptyset 3 \rightsquigarrow \emptyset 2 \emptyset 2^3 \emptyset^\infty, \\ &\emptyset^3 3 \rightsquigarrow \emptyset 2 \emptyset 2 \emptyset^\infty, \quad \textit{and} \quad \emptyset^\infty \rightsquigarrow (\emptyset 2)^\infty. \end{aligned}$$

Proof. The first case comes from $1 \xrightarrow[\emptyset]{1} .111 \xrightarrow[\emptyset^\infty]{}$. Otherwise, the evolution starts along some path in the following diagram. (On the vertical arrows, inputs are on the left and outputs on the right.)

$$\begin{tikzcd} 1 \arrow[r, "\emptyset", "\emptyset"'] \arrow[d, "3"', "\emptyset"] & 11 \arrow[r, "\emptyset", "2"'] \arrow[d, "3"', "2"] & 1 \arrow[r, "\emptyset", "\emptyset"'] \arrow[d, "3"', "\emptyset"] & 11 \arrow[r, "\emptyset^\infty", "(2\emptyset)^\infty"'] \arrow[d, "3"', "2"] & {} \\ 113 & .13 \arrow[d, "\emptyset^\infty"] & 113 & .13 \arrow[d, "\emptyset^\infty"] & \\ & {} & & {} & \end{tikzcd}$$

Considering the possible continuations from the occurences of the state 113 completes the proof. ☐

Lemma 4.10. *If $B_i = 11$, we have*

$$\begin{aligned} & 3 \rightsquigarrow 2\emptyset^\infty, \quad 1^\infty \rightsquigarrow 2^\infty \quad \textit{and} \quad 1^a\emptyset^\infty \rightsquigarrow 2^{a+1}(\emptyset 2)^\infty \quad \textit{for } 0 \le a \le 2; \\ & 1^a\emptyset 31\emptyset \rightsquigarrow 2^{a+1}\emptyset 2^7\emptyset^\infty \quad \textit{and} \quad 1^a\emptyset 311 \rightsquigarrow 2^{a+1}\emptyset 2^4\emptyset^\infty \quad \textit{for } a = 0 \textit{ or } 1; \\ & 1^a\emptyset 3\emptyset^b 3 \rightsquigarrow 2^{a+1}\emptyset 2^{b+2}\emptyset^\infty \quad \textit{for } 0 \le a \le 2 \textit{ and } 1 \le b \le 3; \\ & 1^3\emptyset 3\emptyset 3 \rightsquigarrow 2^4\emptyset 2^3\emptyset^\infty \quad \textit{and} \quad 1^a\emptyset 3\emptyset^\infty \rightsquigarrow 2^{a+1}\emptyset 2^\infty \quad \textit{for } 0 \le a \le 2; \\ & 1^a\emptyset^2 3 \rightsquigarrow 2^{a+1}\emptyset 2\emptyset^\infty \quad \textit{for } 0 \le a \le 2; \\ \textit{and} \quad & 1^a\emptyset^3 3 \rightsquigarrow 2^{a+1}\emptyset 2\emptyset 2^3\emptyset^\infty \quad \textit{for } a = 0 \textit{ or } 1; \end{aligned}$$

Proof. The first case is given by $11 \xrightarrow[2]{3} \mathbf{.}13 \xrightarrow[\emptyset^\infty]{}$. Noting that $11 \xrightarrow[2]{1} 11$, the second is clear, and all the rest come from an evolution starting $11 \xrightarrow[2^a]{1^a} 11 \xrightarrow[2]{\emptyset} 1$ for $0 \le a \le 3$. ☐

Lemma 4.11. *For $1 \le i < n$ and any $B_i \in \mathcal{S}^2_{\{1,3\}}$, s_{i+1} is one of the following streams.*

$$\begin{array}{ll} 2^a(\emptyset 2)^\infty & \textit{for } 0 \le a \le 3; \\ \emptyset^a 2^\infty & \textit{for } 0 \le a \le 2; \\ 2^a\emptyset 2^\infty & \textit{for } 1 \le a \le 3; \\ 2^a\emptyset^\infty & \textit{for } 0 \le a \le 7; \\ 2^a\emptyset 2^b\emptyset^\infty & \textit{for } 0 \le a \le 2 \textit{ and } b = 1,3,4,5 \textit{ or } 7; \\ 2^3\emptyset 2^a\emptyset^\infty & \textit{for } a = 1,3,4 \textit{ or } 5; \\ \emptyset 2\emptyset 2^a\emptyset^\infty & \textit{for } a = 1,3,4 \textit{ or } 7; \\ \emptyset^a 2^4\emptyset 2^3\emptyset^\infty & \textit{for } 0 \le a \le 2; \\ 2^a\emptyset 2\emptyset 2^3\emptyset^\infty & \textit{for } a = 1 \textit{ or } 2. \end{array}$$

Proof. This is just a matter of checking that all the values of s_{i+1} from Lemmas 4.4–4.10 appear in the above list. ☐

In fact, not all the above streams can occur.

Lemma 4.12. *For $1 \le i < n$ and any $B_i \in \mathcal{S}^2_{\{1,3\}}$, s_{i+1} is not equal to $2^a\emptyset 2^5\emptyset^\infty$ for $0 \le a \le 3$, $2^3\emptyset 2^4\emptyset^\infty$, or $2^2\emptyset 2\emptyset 2^3\emptyset^\infty$.*

Proof. From Lemmas 4.4–4.10, these streams could only arise as s_{i+1} if B_i and an initial portion t of t_i were as follows.

s_{i+1}	B_i	t
$\emptyset 2^5\emptyset^\infty$	1	$3\emptyset^3 3$
$2^a\emptyset 2^5\emptyset^\infty$, $1 \le a \le 3$	11	$1^{a-1}\emptyset 3\emptyset^3 3$
$2^3\emptyset 2^4\emptyset^\infty$	11	$1^2\emptyset 3\emptyset^2 3$
$2^2\emptyset 2\emptyset 2^3\emptyset^\infty$	11	$1\emptyset^3 3$

In all but one case, t_i contains $3\emptyset^3 3$ or $1\emptyset^3 3$, which implies that s_i contains $2\emptyset^2 2$; but this is impossible by the previous lemma. In the remaining case, $t = 1^2\emptyset 3\emptyset^2 3$, which implies that s_i starts with $\emptyset 2^3\emptyset 2$, which is also impossible. $\square$

We let $\mathcal{S}_{\{\emptyset,2\}}$ be the set of streams appearing in Lemma 4.11 but not in Lemma 4.12.

Theorem 1. *Every mature string has longevity at most 20.*

From Lemma 2.4, this implies Guy's version of the Cosmological Theorem.

Corollary 1. *Every digit string has longevity at most 24.*

Proof of Theorem 1. Let the state of $\mathbb{M}$ after k inputs (all $\emptyset$) have been consumed be $\big(C^{(k)}, A_n^{(k)}, B_{n-1}^{(k)}, A_{n-1}^{(k)}, \ldots, B_1^{(k)}, A_1^{(k)}\big)$, so that

$$\alpha^k(M) = C^{(k)} A_n^{(k)} B_{n-1}^{(k)} A_{n-1}^{(k)} \cdots B_1^{(k)} A_1^{(k)}.$$

Let $\mathbb{N}_i$ be the flattening of $\mathbb{B}_i\mathbb{A}_i$ for $1 \le i < n$, and of $\mathbb{C}\mathbb{A}_n$ for $i = n$. Give $\mathbb{N}_n \cdots \mathbb{N}_1$ the initial state corresonding to M and input $\emptyset^\infty$, and let its state after k inputs be $\big(N_n^{(k)}, \ldots, N_1^{(k)}\big)$, so that also $\alpha^k(M) = N_n^{(k)} \cdots N_1^{(k)}$. $\mathbb{N}_i$ has input s_i for $1 \le i \le n$ and output s_{i+1} for $1 \le i < n$. Let $1 \le i < n$. Since $s_{i+1} \in \mathcal{S}_{\{\emptyset,2\}}$, $s_{i+1} = u_i v_i^\infty$, where $\mathrm{length}(u_i) = 10$ and $v_i = \emptyset, \emptyset 2, 2\emptyset$ or 2. If $v_i = \emptyset$, $\alpha^{10}(M)$ splits as

$$N_n^{(10)} \cdots N_{i+1}^{(10)} \mathbf{.} N_i^{(10)} \cdots N_1^{(10)}.$$

Streams ending in $(\emptyset 2)^\infty$ only arise in Lemmas 4.9 and 4.10, and it follows that if $v_i = \emptyset 2$ or $2\emptyset$ then $B_i^{(10)} = 1$ or 11 and t_i contains only $\emptyset$ after its first 10 terms. This implies that $A_i^{(10)} = 22$ and s_i contains only $\emptyset$ after its first 10 terms, and it follows that $\alpha^{10}(M)$ splits as

$$N_n^{(10)} \cdots N_{i+1}^{(10)} (1 \text{ or } 11)\mathbf{.}22\mathbf{.}N_{i-1}^{(10)} \cdots N_1^{(10)}.$$

Now suppose that $v_i = 2$. Examining the proofs of Lemmas 4.4–4.10, we see that this can arise in three ways. First, we may have $B_i^{(10)} = 112R$ with R as in Lemma 3.2. In this case, $\alpha^{10}(M)$ splits as

$$N_n^{(10)} \cdots N_{i+1}^{(10)} 112\mathbf{.}R A_i^{(10)} N_{i-1}^{(10)} \cdots N_1^{(10)}.$$

Second, we may have $B_i^{(10)} = 113$ and t_i containing only $\emptyset$ after its first 10 terms. This implies that $A_i^{(10)} = 22$ and s_i contains only $\emptyset$ after its first 10 terms, and it follows that $\alpha^{10}(M)$ splits as

$$N_n^{(10)} \cdots N_{i+1}^{(10)} 113\mathbf{.}22\mathbf{.}N_{i-1}^{(10)} \cdots N_1^{(10)}.$$

Finally, we may have $B_i = 11$ and $t_i = 1^\infty$. This implies that $A_i = 2$ and $s_i = \emptyset^\infty$, and it follows that $\alpha^{10}(M)$ splits as

$$N_n^{(10)} \cdots N_{i+1}^{(10)} 112\mathbf{.}N_{i-1}^{(10)} \cdots N_1^{(10)}.$$

In all cases, we may write $N_i^{(10)} = L_i R_i$, where $L_i = \emptyset, 1$ or 11 if $v_i = \emptyset, \emptyset 2$ or $2\emptyset$, respectively, and $L_i = 112$ or 113 if $v_i = 2$, and $\alpha^{10}(M)$ splits as

$$N_n^{(10)} \cdots N_{i+1}^{(10)} L_i\mathbf{.}R_i N_{i-1}^{(10)} \cdots N_1^{(10)}.$$

Setting $L_0 = \emptyset$, we have the splitting

$$N_n^{(10)} L_{n-1} \centerdot R_{n-1} L_{n-2} \centerdot \cdots \centerdot R_1 L_0$$

of $\alpha^{10}(M)$, so it suffices to show that all the factors in this splitting have longevity at most 10. There are only finitely many possibilities for these factors and they may be obtained as follows. For $j = 1$ or 2, let $\mathcal{X}_j$ be the set of triples (S, s, L), where $S = (S_1, S_2) \in \mathcal{S}^j_{\{1,3\}} \times \{2, 22, 222\}$, $S_1 S_2$ looks 4 days old, $s \in \mathcal{S}_{\{\emptyset,2\}}$, and, writing $s = uv^\infty$ with $\mathrm{length}(u) = 10$, $L = \emptyset, 1$ or 11 if $v = \emptyset, \emptyset 2$ or $2\emptyset$, respectively, and $L = 112$ or 113 if $v = 2$. Let $\mathcal{Y} \subset \mathcal{X}_1$ consist of those triples with $L \neq 112$. For $(S, s, L) \in \mathcal{X}_1$, let $N(S, s)$ be the state of $\mathbb{CA}$, starting from S, after the first 10 inputs have been taken from s. Then $N_n^{(10)} L_{n-1} = N(S, s)L$ for some (S, s, L). Similarly,for $(S, s, L) \in \mathcal{X}_2$, let S' be the state of $\mathbb{BA}$, starting from S, after the first 10 inputs have been taken from s. If the first atom of S' is 1, 11, 112 or 113, let $R(S, s)$ be the remainder of S', and otherwise let $R(S, s) = S'$. Then, for $1 \leq i < n$, $R_i L_{i-1} = R(S, s)L$ for some (S, s, L). For $(S, s, L) \in \mathcal{X}_2$, $N(S, s)$ and $R(S, s)$ are both defined, and $N(S, s)L = T \centerdot R(S, s)L$ for some string T, so it is enough to show that $N(S, s)L$ has longevity at most 10 for all $(S, s, L) \in \mathcal{X}_1$. If s ends in 2^∞, the longevity of $N(S, s)112$ is at most that of $N(S, s)113$, since for any common atom ending 13, replacing the final 3 by 2 gives a common atom. Hence we need only consider $(S, s, L) \in \mathcal{Y}$. Unfortunately, there are still many cases. Of the 37 strings in $\mathcal{S}^1_{\{1,3\}}$, 19 cannot be followed by 2^3 in a string that looks 4 days old, and 8 others not by 2^1, so there are 84 choices for S. Since there are 40 streams in $\mathcal{S}_{\{\emptyset,2\}}$, we have $|\mathcal{Y}| = 84 \times 40 = 3360$. The corresponding strings $N(S, s)L$ are computed, and their longevities checked, by the program `proof2` referred to earlier, completing the proof. $\square$

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