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Architectural Applications of Graph Theory

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1. Introduction

Any design problem may be posed as a problem on the construction of an artifact which has a set of specified functional properties. Some design theorists emphasize the functional properties, and use them to specify a set of variables which define a solution space. This space is then searched for a suitable design. In this chapter we take the point of view that a knowledge of the possible structures or forms which the designs may take is important. If these possible structures can be generated, then they may be subjected to analysis to determine their functional properties.

In architectural design, a central problem concerns the synthesis of floor plans. Once a designer has a set of possible floor plans answering the client's brief, he may then subject these to analytical tests as part of a selection process. Following this, a smaller number of candidate designs may be evaluated before a single design is adopted. In practice, such a procedure tends to be iterative and cyclical. However, at the first stages in design, how is the set of possible floor plans arrived at? Alternatively, where do such designs come from?

Before the advent of modern systems theory, such questions held little interest, and gave rise to commonplace answers. Out of an "infinite" number of possibilities, the designer would have chosen a set of suitable possibilities through experience and intuition. However, the systems designer is now inclined to ask: can I find production rules to generate all possible designs (of a certain kind), and can I enumerate and classify these designs in such a way as to help my search in the selection of preferred designs? With computer aids, these questions may be answered for the designer through the use of combinatorial methods.

In the initial stages, designs are combinatorial objects or pure structures. The potential designs at these initial stages can often be exhaustively enumerated in a practical sense. Such designs mature during the design process as they acquire more character through spatial, physical and material transformation. The multifarious effect of these transformations leads the traditional designer to refer to "infinite possibilities". The study of transformations is central to morphological enquiries in architecture, but this must rest on an investigation of premorphology or the study of fundamental architectural forms. In this chapter we investigate the premorphology of floor plans, and demonstrate certain types of transformations related to shape and size. This is an exercise in the theory of planar maps, linear graphs and networks.

Graph theory has been used in previous approaches to floor plan synthesis, although no systematic presentation has been given which defines a fundamental set of plans and their subsequent transformations. Levin [20] introduced a graph whose vertices represent activity areas or spaces in the plan and whose edges represent adjacency or contiguity of the spaces, and he demonstrated this graph to be a useful representation in spatial allocation problems current at the time. Cousin [9] developed the adjacency graph representation, but in common with Levin, neither the possible plane embeddings of the adjacency graphs, nor the problem of realizing this adjacency structure as a floor plan, are dealt with very clearly.

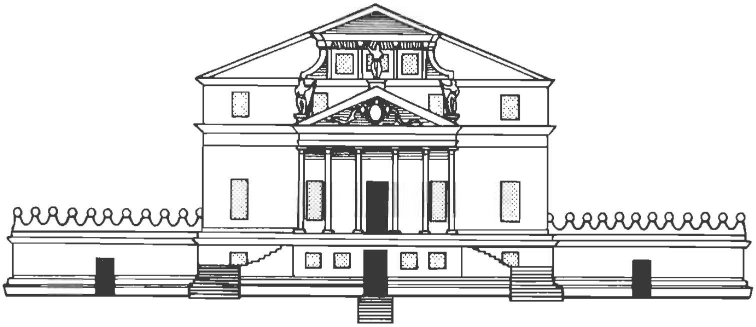
Grason [14] also used an adjacency graph, restricting the realizations of the adjacency structure to be rectangular dissections – that is, plans in which each room is rectangular, as is the boundary of the plan. Teague [30] similarly restricted the type of plan, but represented the rectangular spaces by arcs in a network. Mitchell *et al.* [25] generated the sets of rectangular dissections for small numbers of rooms (see also [12]), and it was from this set that they could choose designs for small house plans; they took the rectangular dissections as their initial or basic set of forms. In the present chapter, however, we consider the rectangular dissections as a set derived by transformation from a more general set of fundamental architectural forms.

2. Floor Plan Arrangements

A **floor plan** is a finite set of walls in the plane. These walls are of many types according to their detailed architectural properties, which typically include the presence of doors and windows, the thermal and load-bearing properties, and a variety of other features depending on the context. A plan is constructed to allow a set of activities to be pursued within a given area. In floor plans, particularly in domestic dwellings, it is often necessary for individual activity areas to be enclosed by walls. In open plan schemes, however, the walls or partitions define the activity areas, but they do not necessarily define enclosed regions. These open plan schemes are derived from plans with walls enclosing the activity areas by the removal of certain walls. Figure 1 shows the Villa Malcontenta by Palladio, in which the walls clearly define enclosed regions, whereas Fig. 2 shows the Farnsworth House by Mies van der Rohe, which is of the open plan type (see also Fig. 16).

Although the type, shape, length and orientation of the walls are exactly specified in a plan, it is our purpose (in this and the next two sections) to examine the possible arrangements of walls as regards only their incidence and relative disposition in the plane. For example, although the walls of the plans shown in Fig. 3 have different lengths and orientations, we consider them as having the same arrangements. Our aim is to investigate the underlying structures of floor plans, and the notions of planar and plane maps are particularly relevant for this purpose.

A **planar map** is a connected graph, embedded on the sphere and separating the remainder of the surface into a finite number of simply connected regions. Two planar maps are called **equivalent** if there is a



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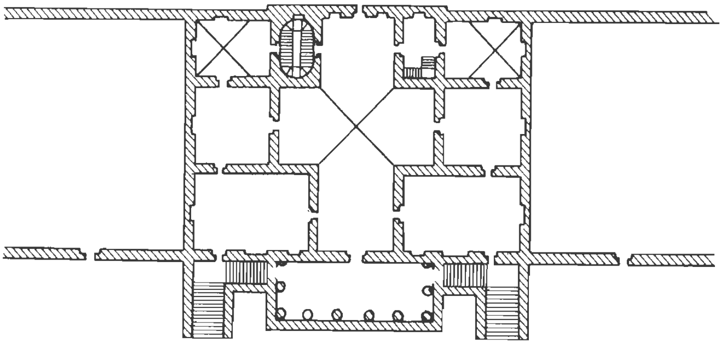

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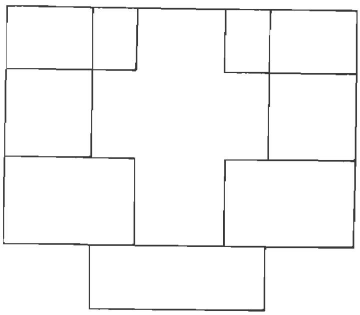

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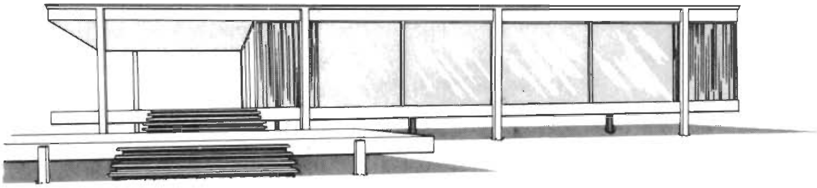
Fig. 1



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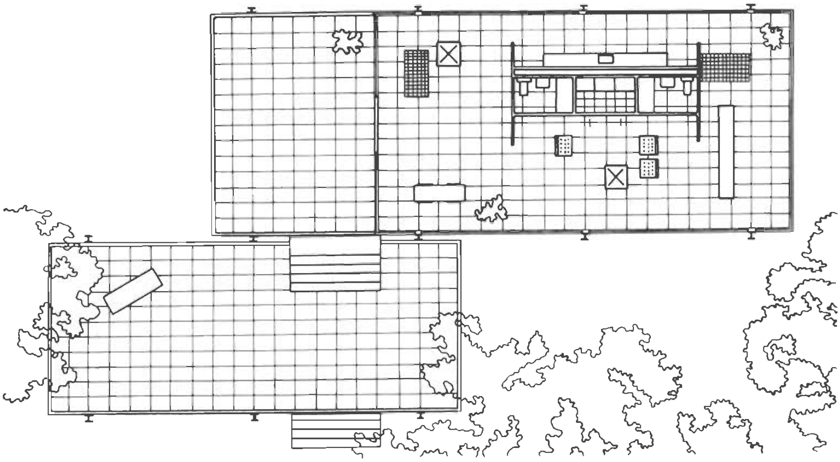

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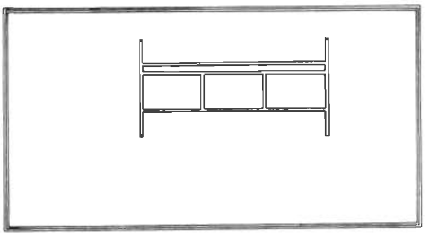

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Fig. 2

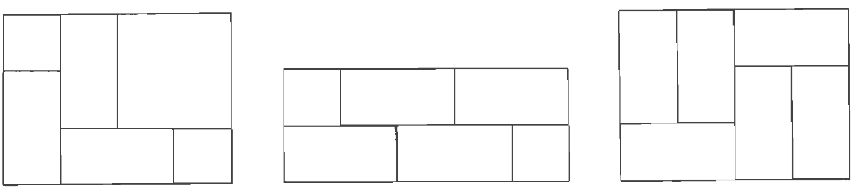

Fig. 3

homeomorphism of the sphere onto itself taking one map into the other. Planar maps with one labeled region are equivalent if there is a homeomorphism of the sphere taking one map into the other, and preserving the labeled region. Distinct planar maps with a labeled region correspond to distinct arrangements of walls in floor plans, with the edges corresponding to the walls, and the labeled region to the outside. We consider only those floor plans with such arrangements. This includes the assumption that the walls form a connected graph. In general, the individual components of a disconnected floor plan may be treated as separate entities, whose arrangement presents a site layout problem rather than a floor plan problem. We note that the difference between such problems lies not so much in the type of underlying structure, as in the interpretation of that structure. Indeed, the analysis presented in this chapter may be used (with suitable interpretation) to address problems from the arrangement of furniture in an office to the layout of a city.

A planar map is **rooted** (see [33]) by directing an edge R and distinguishing one side of R. The negative end of R is the **root vertex**, R is itself the **root edge**, and the region on the distinguished side of R is the **root region**. Two rooted planar maps are **equivalent** if there is a homeomorphism of the sphere onto itself, taking one into the other, and preserving the root vertex, root edge and root region. Tutte [34] noted that rooting a map destroys its symmetry, and in view of this, it is much easier to enumerate rooted planar maps than unrooted ones. A rooted planar map represents a floor plan arrangement with the root region as the outside region, the root edge as the "facade", and a "handedness" given by the direction of the root edge.

If a planar map with a labeled region is embedded in the plane with all of its regions finite except for the labeled region, then we obtain a **plane map**, and such maps clearly provide a natural representation of floor plan arrangements. We root a plane map by directing an edge in the boundary of the infinite region.

Any geometric dual of a plane map is another plane map representing a floor plan arrangement, but now in terms of the adjacencies between regions. We note that a region can be adjacent to itself, and that regions can be multiply adjacent. For each plane map there are, in general, many dual plane maps, but to each rooted plane map there corresponds a unique rooted dual plane map in an obvious way.

Plane maps thus comprise a description or representation of floor plans, considered as arrangements of walls. However, the architectural design problem of floor plan synthesis often starts with a set of adjacency or proximity requirements between various activities, and the first aim is to construct a suitable arrangement of regions. These arrangements of regions are represented as arrangements of walls, but at this stage we are primarily interested in the regions which these walls define. In other words, it is the broad outline of the plan that is important, rather than the details. Since we now require the walls to define regions, the corresponding plane maps contain no vertices of valency 1 or 2. Figure 4 shows a plane map representing an arrangement of regions, and a plane map with the same arrangement of regions but with extra walls.

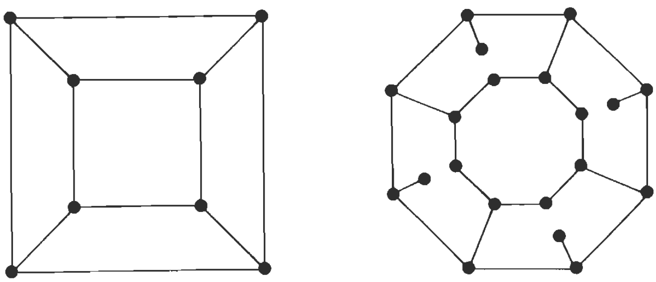

Fig. 4

The plane maps representing arrangements of regions thus have all vertices of valency at least 3. In fact, they are usually trivalent, since walls generally meet at angles not less than a right angle. Although four walls often meet at right angles, we consider this as the limiting case of a pair of trivalent vertices (see Fig. 5).

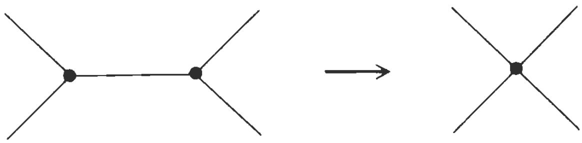

Fig. 5

We now classify the floor plan arrangements represented by trivalent plane maps according to the presence of bridges and 2-edge cuts in the map, where a **2-edge cut** is a pair of edges whose removal disconnects the map. A bridge in the map represents either a single wall joining two parts of the plan, or a single room which forms a ring (see Fig. 6). A 2-edge cut represents either a "through room" in the plan, or a pair of rooms forming a ring (see Fig. 7). The existence of a through room or corridor is certainly a pertinent architectural feature.

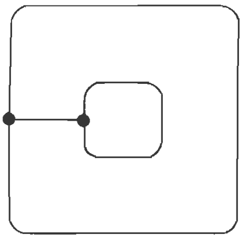

Fig. 6

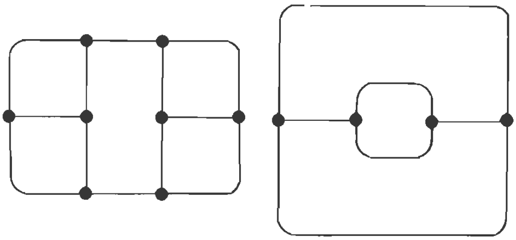

Fig. 7

Trivalent maps with neither a bridge nor a 2-edge cut are called **3-connected**. This terminology is justified since, for trivalent maps with more than three edges, 3-edge-connectedness and 3-connectedness are equivalent. The set of 3-connected trivalent plane maps is particularly useful in approaching the problem of floor plan synthesis, since they represent (in a sense to be made clear in Section 4) a basic set of floor plan arrangements from which all others may be derived by sequences of operations. These operations form the first stages in a process of "ornamentation" by which detailed floor plans are constructed. However, before considering ornamentation we examine the floor plan arrangements represented by trivalent plane maps, and give some relevant enumerative results.

3. The Enumeration of Trivalent Floor Plan Arrangements

In this section we assume that our trivalent maps have no bridges, and let A_r be the number of floor plan arrangements with r rooms, represented by the rooted trivalent plane maps with $r + 1$ regions. The numbers A_r were determined by Tutte [32], who enumerated rooted trivalent planar maps:

$$A_r = \frac{2^{r-1}3!(3r-4)!}{(r-2)!2r!}.$$

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The dual plane maps of trivalent maps are **triangular maps** (see [27]). If a trivalent plane map has no 2-edge cut, then its dual has no multiple edges, and is called a **triangulation**. Let B_r be the number of rooted floor plan arrangements with r rooms, represented by the rooted 3-connected trivalent plane maps with $r + 1$ regions. Then B_r is the number of rooted triangulations with $r + 1$ vertices, as given by Tutte [31]:

$$B_r = \frac{2(4r-7)!}{(r-1)!(3r-4)!}.$$

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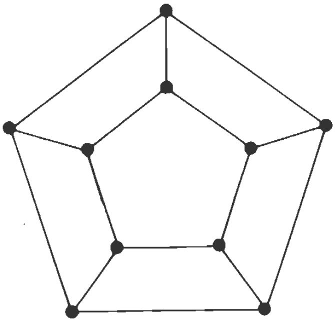
We now consider a class of the 3-connected trivalent plane maps, and we suppose, for the sake of brevity, that the infinite region of a plane map is considered to be a simply-connected region, as it certainly would be if the plane map were considered as a planar map with a labeled region. The class S of 3-connected trivalent plane maps in which no three regions and their mutual edges are multiply connected is particularly important from an architectural point of view, since these maps have realizations in which all regions possess a rectangular boundary. The dual plane maps of members of S are **simple triangulations** – that is, triangulations in which every 3-circuit bounds a region. (A 3-circuit which does not bound a region is often called a "separating triangle".) Conversely, any simple traingulation is the dual of a map in S. Tutte [31] enumerated rooted simple triangulations, thus determining the number of rooted versions of maps in S.

A closed bounded region in the plane divided into triangular regions with $s + 3$ vertices on the boundary and r internal vertices is said to be a triangular map of type $[r, s]$. If there are no multiple edges, it is an **$[r, s]$-triangulation** – that is, a triangulation of an $(s + 3)$-gon with r internal vertices. If no interior edge is incident with two external vertices, then we have a **strong triangulation**, and if it also contains no separating triangle, then it is a **simple triangulation**.

If M^* is any dual of a trivalent map M, in which the vertex v^* corresponds to the infinite region of M, then $M^* - v^*$ is called the **weak dual plane map**. Figure 8 illustrates a trivalent map and its weak dual.



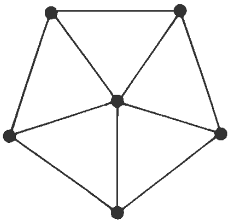


Fig. 8

Each trivalent plane map has a weak dual whose blocks are single edges or general $[r, s]$-triangular maps, and conversely, each such map is the weak dual of a unique trivalent map. We note that cut-vertices in the weak dual represent "through rooms" in the floor plan arrangement. Let $C_{r,s}$ be the number of rooted floor plan arrangements without through rooms, in which there are r internal rooms and $s + 3$ rooms adjacent to the outside. Then $C_{r,s}$ is the number of rooted $[r, s]$-triangular maps (see [27]), and

$$C_{r,s} = \frac{2^{r+1}(2s+3)!(3r+2s+2)!}{(s+1)!^2 r!(2r+2s+4)!}.$$

The weak duals of the 3-connected trivalent maps are $[r, s]$-triangulations. The number $R_{r,s}$ of $[r, s]$-triangulations, although not given explicitly by Brown [6], may be evaluated from his results. $R_{r,s}$ is the number of floor plan arrangements represented by the 3-connected trivalent maps, with r internal rooms and $s + 3$ rooms adjacent to the outside (see Table I).

Table I. Values of $R_{r,s}$

s	r = 0	1	2	3	4	5	6	7
0	1	1	1	4	16	78	457	2938
1	1	2	5	18	88	489	3071	
2	1	4	14	69	396	2503		
3	3	11	53	295	1867			
4	4	28	178	1196				
5	12	91	685					
6	27	311						
7	82							

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We may also derive the number R_n of such arrangements with a total of n rooms (see Table II).

Table II.

n	3	4	5	6	7	8	9	10
R_n	1	2	4	16	63	328	1933	12653

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The number $J_{r,s}$ of $[r, s]$-triangulations with reflection symmetry may also be evaluated from Brown's results. $J_{r,s}$ is the number of floor plan arrangements, represented by the 3-connected trivalent maps which have reflection symmetry, and with r internal rooms and $s + 3$ rooms adjacent to the outside (see Table III).

Table III. Values of $J_{r,s}$

s	r = 0	1	2	3	4	5	6	7
0	1	1	1	3	8	23	68	215
1	1	2	4	10	29	86	266	
2	1	3	7	19	57	176		
3	2	6	18	52	166			
	2	8	26	82				
	5	18	68					
6	5	23						
7	14							

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We may also derive the number J_n of such arrangements with a total of n rooms (see Table IV).

Table IV.

A5028 (correct)

n	3	4	5	6	7	8	9	10
J_n	1	2	4	12	33	102	312	~~1006~~ 1010

A169807 (erroneous)

We note that for relatively large numbers of rooms, only a small proportion of the floor plan arrangements, as represented by the 3-connected trivalent maps, have reflection symmetry. This gives some indication of the restriction on possible arrangements which is imposed if an axis of symmetry is required.

4. Fundamental Architectural Plans and Ornamentation

In Section 2 we referred to the 3-connected trivalent plane maps as representing a basic set of plans. However, since they are the result of labeling one region in the corresponding planar maps, we can take the set of 3-connected trivalent planar maps as representing the fundamental architectural plans (see [22]), and the labeling of a region as the first step in ornamentation. For an alternative picture of fundamental plans we use Steinitz's theorem (see [15]), which states that a planar graph is realizable as a 3-polytope if and only if it is 3-connected. Thus every fundamental plan (except the plan with three regions) may be represented by a simple 3-polytope – that is, by a polytope all of whose vertices have valency 3.

The use of ornamentation operations emphasizes the constructive side of the problem of floor plan synthesis, and it is evidently important that the fundamental plans have simple rules for their construction. In fact, any trivalent 3-connected map with $n + 1$ regions may be generated from one with n regions by applying one of the three rules shown in Fig. 9.

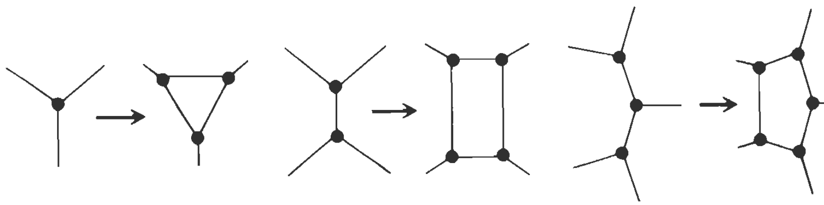

Fig. 9

The number F_n of fundamental architectural plans with n regions was determined by Grace [13], who constructed and enumerated the simple 3-polytopes with $n \leqslant 12$ (see Table V). Bowen and Fisk [13] equivalently enumerated their duals – that is, the triangulations of the surface of a sphere.

Table V.

n	3	4	5	6	7	8	9	10	11	12
F_n	1	1	1	2	5	14	50	233	1249	7595

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Fundamental plans with one labeled region (or equivalently, their projections as 3-connected trivalent plane maps) are called **primary plans.** Their weak duals are general $[r, s]$-triangulations, and are called **primary arrangements.** Figure 10 shows the primary plans with up to six rooms. Those marked † have each room adjacent to the outside. They are arranged according to the valency-sequences of their regions.

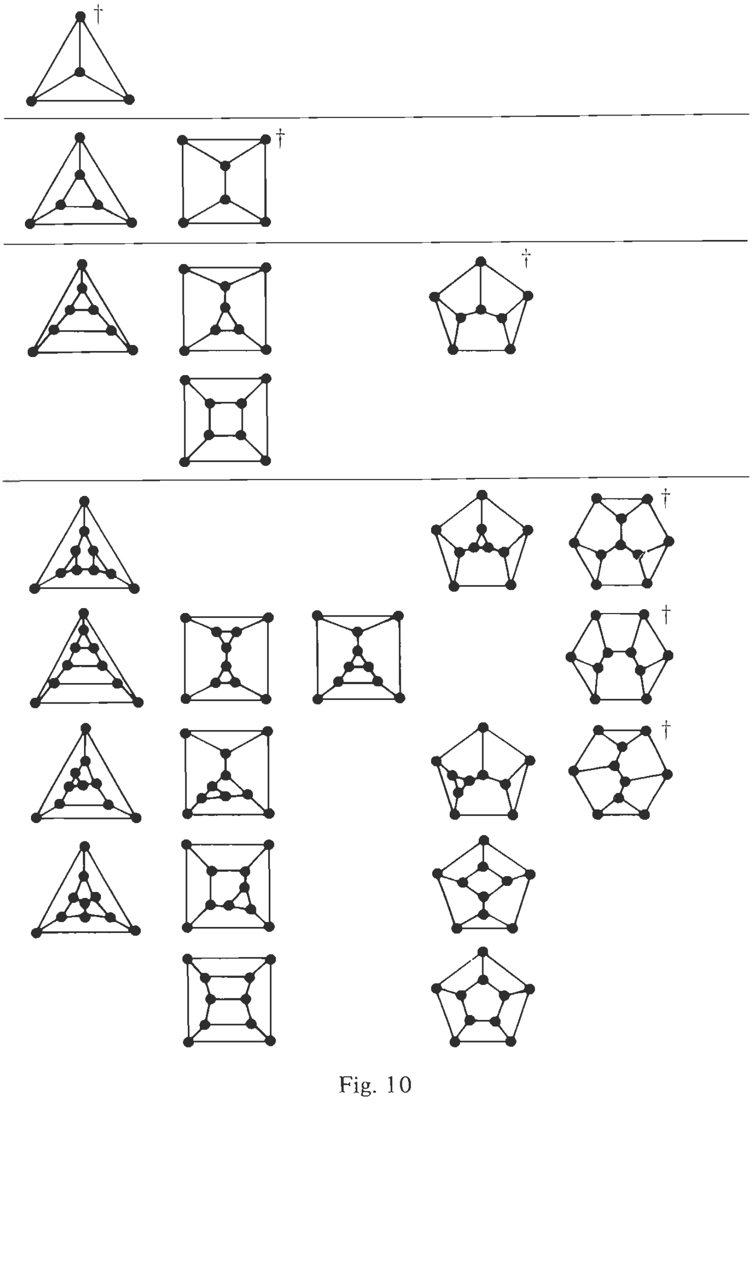

Fig. 10

The main aim of this section is to present a set of operations which ornament the arrangements of walls in the primary plans so that all floor plan arrangements may be obtained. These ornamentation operations modify the architectural features of the floor plan, and are as follows:

(i) An *exchange* is the operation on a trivalent plane map shown in Fig. 11, where the equivalent operation on the dual is also shown. An exchange operation on a 3-connected trivalent map is used to create 2-edge cuts. In an architectural context, its main use is to create through rooms or corridors (see Fig. 12, where the edge involved in the exchange is labeled $\mathcal{E}$).

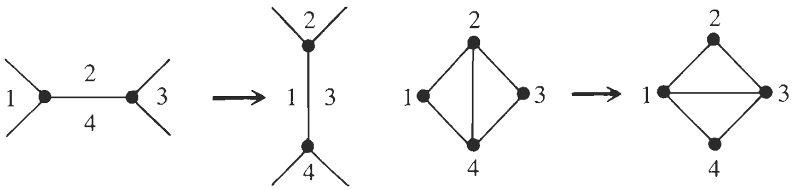

Fig. 11

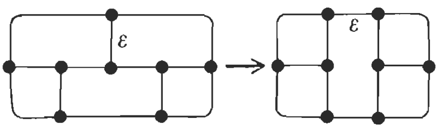

Fig. 12

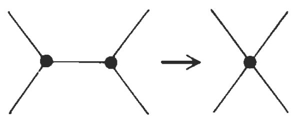

Fig. 13

(ii) An *edge-contraction* (see Fig. 13) removes an adjacency between regions, but brings together regions at a vertex – a feature possibly relevant for the provision of services.

(iii) A *vertex-expansion* is the reverse of an edge-contraction. It can be used to insert vertices of valency 2 into the edges (see Fig. 14), or to construct bridges in the map. We note that an exchange is a combination of an edge-contraction and a vertex-expansion, but since an exchange preserves trivalency, we consider it as a single operation.

(iv) The *addition of edges* which subdivide a specified region is needed when the corresponding activity for which it caters is composed of individual sub-activities. Thus it is possible at the initial stages in floor plan design to consider groups of activities in a single unit, and then, at this ornamentation stage, to differentiate the

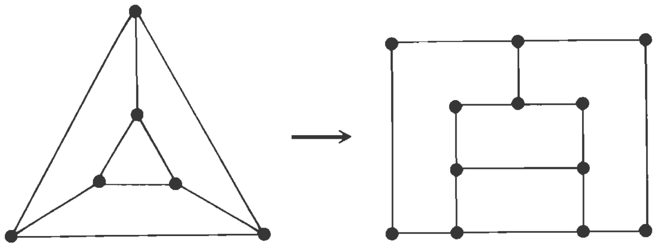

Fig. 14

individual activities. However, the arrangement of the additional walls is really a floor plan problem in its own right, and is often considered as such. A particular form of such ornamentation which deserves mention is the addition of trees planted at a vertex on the boundary of a region (see Fig. 15). The plane embedding of these trees is important, and the number T_r of such planted plane trees with $r \geqslant 1$ edges (see [18]) is given by

$$T_r = \frac{1}{r}\binom{2r-2}{r-1}.$$

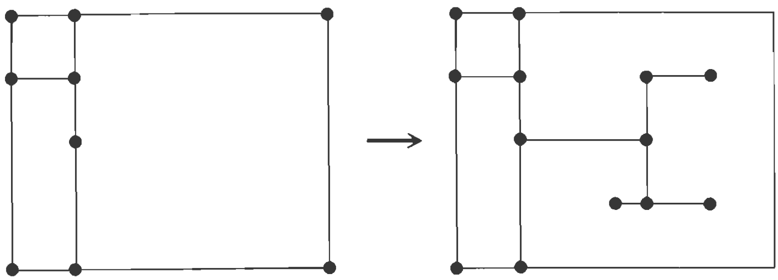

Fig. 15

(v) The *removal of edges* may be considered as a means of producing open plan schemes where the original regions represent distinct activity areas. Figure 16 shows how the Farnsworth House (Fig. 2) may be considered as a floor plan arrangement derived by such ornamentation.

In the next two sections we move away from the underlying structures of floor plan arrangements, and address the problem of their realization under geometric constraints. We consider this realization as a further stage in ornamentation.

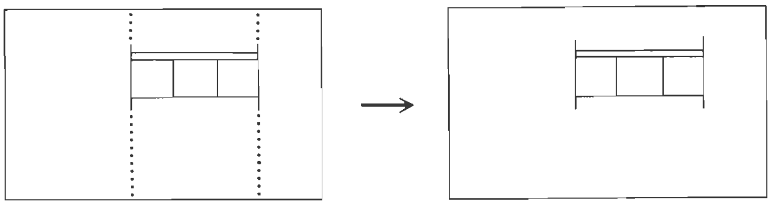

Fig. 16

5. The Shape of Regions in Floor Plan Arrangements

We now investigate the possible realizations of floor plan arrangements represented by the non-separable trivalent plane maps. Let us require that in these realizations all regions are bounded by a rectangle, as is the whole plan. These are called **rectangular dissections**, and many architectural plans have this form; for example, the plans in Fig. 3 are all rectangular dissections. (In what follows we shall ignore the natural abuse of terminology with respect to the vertices of valency 2 at the corners of the plan.) The following theorem of Ungar [35] is important in this context, although it will be improved later in the section.

Theorem 5.1. *Let M be a 3-connected trivalent plane map, in which no three regions and their mutual edges have a multiply-connected union. Then M may be realized by a rectangular dissection.//*

An equivalent formulation of Ungar's theorem is the following corollary:

Corollary 5.2. (*i*) *Each simple triangulation is the dual plane map of a rectangular dissection;*
(*ii*) *each simple triangulation of a polygon is the weak dual of a rectangular dissection.//*

Note that some dissections have two rectangles and the outside region with multiply-connected union, some have 2-edge cuts corresponding to corridors or "through rooms", and others do not correspond to trivalent maps because of the presence of four-way points (see Fig. 17).

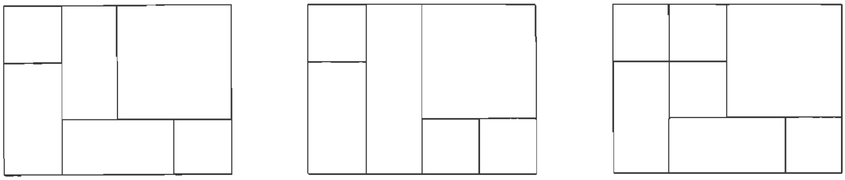

Fig. 17

In what follows, unless otherwise stated, a rectangular dissection is assumed not to have four-way points. In order to obtain the set of all plane maps corresponding to the rectangular dissections we define a special dual plane map – its **augmented dual**, in which the outside region is divided into four parts (see Fig. 18).

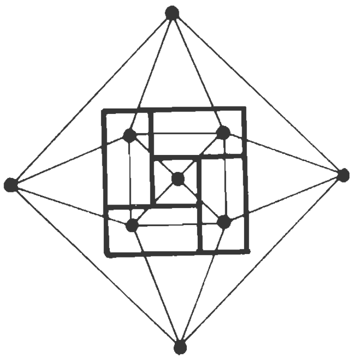

Fig. 18

The augmented duals are simple $[r, 1]$-triangulations, since no three rectangles form a ring, thus precluding a separating triangle in the dual. We may now state a theorem specifying those trivalent maps which are realizable as rectangular dissections. Its corollaries then make this more explicit.

Theorem 5.3. *A plane map is the augmented dual of a rectangular dissection if and only if it is a simple* $[r, 1]$*-triangulation.*

Proof. ⇒ The augmented dual of any rectangular dissection is a simple $[r, 1]$-triangulation.

⇐ The proof is by induction on r. Suppose that each simple $[r, 1]$-triangulation is the augmented dual of a rectangular dissection for $r \leqslant n$, and let T_{n+1} be a simple $[n + 1, 1]$-triangulation. We call a 4-circuit "non-trivial" if it has at least one internal vertex. Let v be an external vertex in T_{n+1}. Then there are two cases to consider:

Case 1. There exists an edge vw which does not belong to a non-trivial 4-circuit. Contracting the edge vw gives a simple $[n, 1]$-triangulation, which (by hypothesis) is the augmented dual of a dissection. A dissection with augmented dual T_{n+1} may then be obtained by an operation of the type shown in Fig. 19.

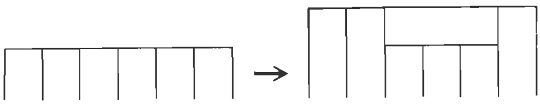

Fig. 19

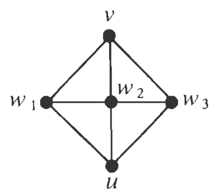

Fig. 20

Case 2. If every edge incident to v belongs to a non-trivial 4-circuit, then it can easily be shown that the configuration shown in Fig. 20 occurs. Since T_{n+1} is planar and simple, the edge w_1w_2 does not belong to a non-trivial 4-circuit. Contracting this edge gives a simple $[n, 1]$-triangulation, which, by hypothesis, is the augmented dual of a dissection. A dissection with augmented dual T_{n+1} may then be obtained by one of the operations shown in Fig. 21.

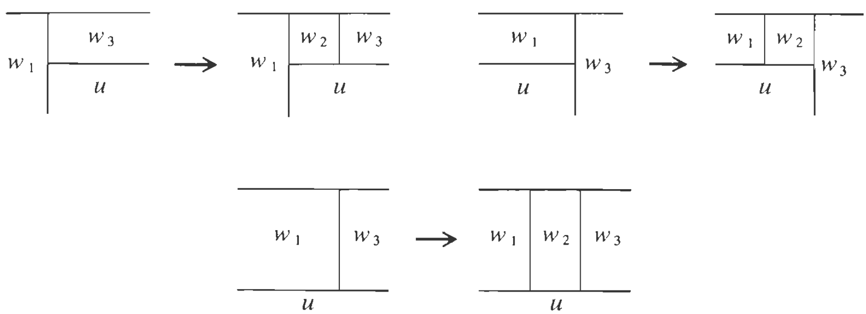

Fig. 21

Finally the simple [1, 1]-triangulation is the augmented dual of a single rectangle. The induction holds and the proof is complete.//

A diagonal of a triangulation of a polygon is an internal edge joining two external vertices on the boundary polygon. To each diagonal there corresponds a pair of 2-vertex components, which we call the **diagonal components.** A diagonal component is **simple** if it is itself a simple triangulation of a polygon. Figure 22 shows a triangulation of a polygon with diagonal vw, and the corresponding diagonal components.

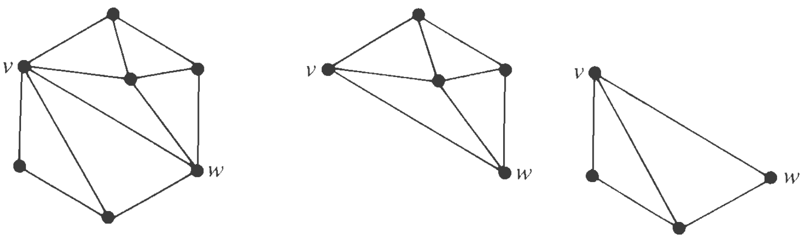

Fig. 22

Corollary 5.4. *A plane map is a weak dual of a rectangular dissection with more than three component rectangles and with no "through rooms" if and only if it is an* $[r, s]$*-triangulation* ($s \geqslant 1$) *without separating triangles and with at most four simple diagonal components.*//

We now consider the generalization to dissections containing "through rooms". Let $\mathscr{M}$ be the class of plane maps which satisfy the following conditions:

(i) each block is either a single edge, a triangle, or an $[r, s]$-triangulation ($s \geqslant 1$) without separating triangles;
(ii) each cut-vertex is contained in exactly two blocks, and no block contains more than two cut-vertices.

Figure 23 shows a typical member of $\mathscr{M}$ with blocks $T_0, \ldots, T_k$ and cut-vertices $c_0, c_1, \ldots, c_{k-1}$.

In T_i ($i = 1, 2, \ldots, k-1$), two external vertices v and w are said to be **close** if at least one of them is a cut-vertex, or if there is a path

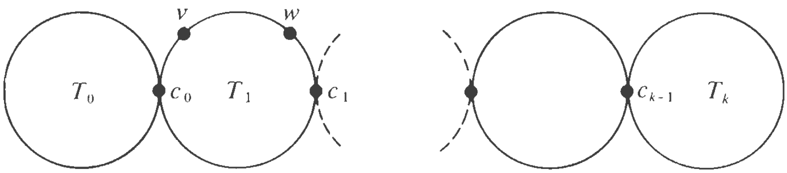

Fig. 23

joining v and w consisting entirely of external edges of T_i and not passing through a cut-vertex; for example, the vertices v and w in Fig. 23 are close. The following corollary completes the characterization of those plane maps which may be realized as rectangular dissections:

Corollary 5.5. *A plane map M is the weak dual of a rectangular dissection if and only if the following conditions hold:*

(*i*) $M \in \mathscr{M}$;
(*ii*) *no pair of close external vertices are joined by an internal edge;*
(*iii*) c_{i-1} *is not joined to* c_i (*for* $i = 1, 2, \ldots, k-1$) *unless* T_i *is a single edge;*
(*iv*) *in* T_0 *and* T_k, *either there are three simple diagonal components one of which contains* c_0 *or* c_{k-1} *respectively, or there are at most two simple diagonal components.//*

If a **rectangular arrangement** is defined to be a close packing of rectangles, then the next result follows from Theorem 5.3:

Theorem 5.6. *A plane map is the weak dual of a rectangular arrangement without four-way points if and only if its blocks are single edges, triangles, or* $[r, s]$*-triangulations* ($s \geqslant 1$) *without separating triangles.//*

If a rectangular dissection is oriented by labeling the four sides of the boundary rectangle N, S, E and W, then the augmented dual has the corresponding external vertices labeled. This is equivalent to rooting the triangulation with root vertex W and root edge WN (see Fig. 24).

A coloring of the edges of a rooted simple $[r, 1]$-triangulation with two colors (x and y, say) is called a **valid coloring** if it satisfies the following three conditions:

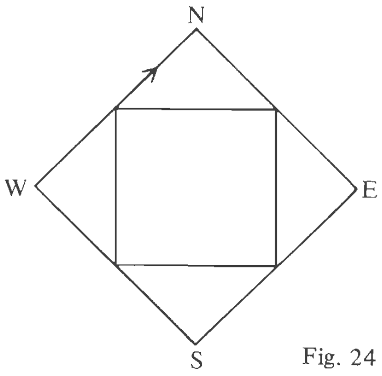

Fig. 24

(i) internal edges incident to N and S are colored x, and those incident to E and W are colored y;
(ii) no 3-circuit has edges of only one color;
(iii) at each vertex, the edges form four groups, two of each color, with each pair separated by a group of the other color.

If the edges of the augmented dual of an oriented dissection are colored x or y according as they represent a "horizontal" (E or W) or "vertical" (N or S) wall, then the resulting coloring is valid. In fact, the converse is also true:

Theorem 5.7. *Each valid coloring of a rooted simple* $[r, 1]$*-triangulation is the augmented dual of an oriented rectangular dissection in which the colors distinguish between the N/S and E/W aligned edges.//*

We can now define oriented rectangular dissections to be **equivalent** if their corresponding colored augmented duals are equivalent.

We have defined the class of trivalent floor plan arrangements which can be realized as rectangular dissections. However, we note that to each arrangement in this class there generally correspond many rectangular dissections. This is because we assign to the arrangement an N, S, E and W setting, and because of the different valid colorings of the corresponding augmented dual. These dissections are considered to be ornamented versions of the relevant floor plan arrangements under the constraint of rectangularity. In the next section we consider further ornamentation of these rectangular dissections, with ornamentation referring to the possible dimensions which the regions in the plan may assume.

6. The Dimensions of Regions in Floor Plan Arrangements

The previous four sections examined possible floor plan arrangements, and the classes which may be realized under the geometrical constraint that all the regions have a rectangular boundary. If those arrangements are to be realized as floor plans, further geometrical constraints concerning the dimensions of the regions must be considered. In the context of floor plan synthesis we require an arrangement not only satisfying certain adjacency constraints, but also realizable with the regions satisfying dimensional constraints.

Given a rooted simple $[r, 1]$-triangulation which represents a suitable arrangement, each valid coloring gives an oriented rectangular dissection. We want to examine this set of dissections to discover which can accommodate a given set of dimensional constraints.

For each oriented rectangular dissection, a network (digraph) can be defined whose arcs represent the rectangles (see Brooks *et al.* [5]). The horizontal sides of the rectangles form a point set whose connected components are the horizontal line segments. We now represent each horizontal line segment by a vertex. Each rectangle has upper and lower sides in distinct horizontal line segments, and is represented by an arc joining the corresponding vertices directed from the upper side to the lower side. The vertices corresponding to the horizontal sides of the boundary rectangle are joined by an arc directed from the lower side to the upper side of the boundary rectangle and representing the outside space. This network is embedded in the plane as a plane map in which the order of the incoming and outgoing arcs at a vertex is the same as the order of the rectangles above and below the corresponding horizontal line segment. The connecting arc lies on the boundary of the infinite region, and is directed counter-clockwise. For each dissection this network is a directed nonseparable plane map; note that it may be derived directly from the validly-colored triangulation. This directed plane map is called the **vertical network**, and the **horizontal network** is defined similarly; in Fig. 25 we show a dissection with the corresponding vertical and horizontal networks. Note that the vertical and horizontal networks of an oriented rectangular dissection are dual plane maps. In what follows, vertical networks are considered, although the horizontal network formulation is equivalent.

We now give some elementary network theory, due to Branin [4].

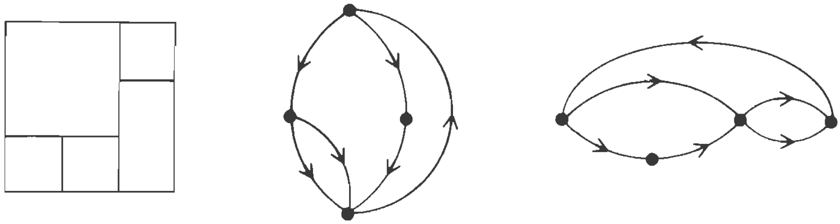

Fig. 25

Consider a directed non-separable plane map with vertex-set $\{v_1, \ldots, v_n\}$ and arc-set $\{e_1, \ldots, e_m\}$, and suppose that e_m lies on the boundary of the infinite region, and is directed from v_n to v_1 with counter-clockwise orientation. We define the $m \times n$ **incidence matrix** $\mathbf{B} = (b_{ij})$, where

$$b_{ij} = \begin{cases} +1, & \text{if } e_i \text{ is incident at } v_j \text{, and oriented towards } v_j; \\ -1, & \text{if } e_i \text{ is incident at } v_j \text{, and oriented away from } v_j; \\ 0, & \text{if } e_i \text{ is not incident at } v_j. \end{cases}$$

The matrix $\mathbf{B}$ contains redundant information since its rows have zero sum, and by deleting a column we obtain an $m \times (n-1)$ edge-vertex matrix $\mathbf{B}_0$.

Given a fundamental set of circuits, we can now define the $m \times (m-n+1)$ **circuit-edge matrix** $\mathbf{C} = (c_{ij})$, where

$$c_{ij} = \begin{cases} +1, & \text{if } e_i \text{ is in the } j\text{th fundamental circuit, and the orientations coincide;} \\ -1, & \text{if } e_i \text{ is in the } j\text{th fundamental circuit, and the orientations do not coincide;} \\ 0, & \text{if } e_i \text{ is not in the } j\text{th fundamental circuit.} \end{cases}$$

We may similarly define the $m \times (n-1)$ cut-set-edge matrix $\mathbf{D}$ and the region-edge matrix $\mathbf{R}$.

Let x_i and y_i denote the "through" and "across" variables on e_i. We can then formulate Kirchhoff's Laws (see Section 3 of Chapter 4) as the following set of linearly independent equations:

$$\begin{aligned} &\mathbf{B}_0^{\mathrm{T}}\mathbf{x} = 0 \quad \text{or} \quad \mathbf{D}^{\mathrm{T}}\mathbf{x} = 0 \\ &\mathbf{C}^{\mathrm{T}}\mathbf{y} = 0, \end{aligned} \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_m)^T$ and $\mathbf{y} = (y_1, \ldots, y_m)^T$. If we also require that

$$x_i \geqslant 0 \text{ and } y_i \geqslant 0, \text{ for } i = 1, 2, \ldots, m-1, \text{ and } x_m \geqslant 0, y_m \leqslant 0, \quad (2)$$

then the solutions of (1) and (2) specify oriented rectangular dissections with x_i and y_i (for $i = 1, 2, \ldots, m-1$) as the horizontal and vertical dimensions of the component rectangles, and x_m and y_m as the dimensions of the boundary rectangle. However, given a dissection and its corresponding network, a solution of (1) and (2) does not necessarily provide a dissection equivalent to the original one, since the adjacency structure will impose certain conditions on the dimensions. In order to examine the possible dimensions the rectangles in a given dissection may assume, it is necessary to solve (1) and (2) under these extra conditions.

If we now impose conditions of the type $a_i \leqslant x_i \leqslant b_i$ and $c_i \leqslant y_i \leqslant d_i$ (for $i = 1, 2, \ldots, m$) on the through and across variables, then the corresponding solutions of (1) and (2) give dissections in which the dimensions of the rectangles are bounded accordingly. Conditions of the form $A_i \leqslant x_i y_i \leqslant B_i$ correspond to area constraints on the rectangles, and we can consider such constraints by defining a new variable $a_i = x_i y_i$ and expressing the Kirchhoff equations in terms of a_i and x_i, or a_i and y_i.

Suppose that a suitable solution to (1) and (2) has been found which satisfies all of the requirements. Then a method for reconstructing the corresponding rectangular dissection is of practical significance, and we may do this elegantly by deriving sets of vertex-quantities $\mathbf{B}^T\mathbf{y}$ and region-quantities $\mathbf{R}\mathbf{x}$. Since each region of the network corresponds to a vertical line segment, and each vertex corresponds to a horizontal line segment, these quantities may be used to define the line segments which specify the dissection. Figure 26 shows a network with a set of flows satisfying (1) and (2), and the same network with the derived vertex- and region-quantities. The values on the vertices and regions represent the distances from horizontal and vertical datum lines respectively.

If we now remove the orientations from all arcs (except the connecting arc) of a network corresponding to a rectangular dissection, then we obtain a rooted non-separable plane map. Conversely, it can be shown that each rooted non-separable plane map may be directed in such a way as to be the network of an oriented rectangular dis-

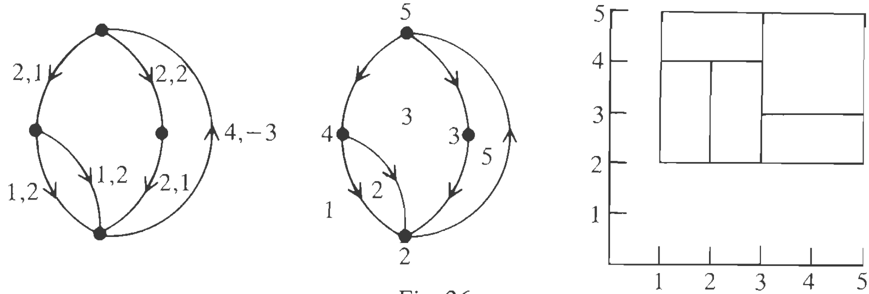

Fig. 26

section. Mullin [27] has noted that the number of non-separable plane maps with $r + 1$ edges is equal to the number of rooted simple $[r, 1]$-triangulations, and it would be interesting if the relationship between both sets of maps and the rectangular dissections provided an explanation.

It is now possible to formulate a procedure by which, given a set of adjacency and dimension requirements, it is possible to derive rectangular dissection solutions to the floor plan problem (if they exist). This procedure may be divided into the following steps:

(i) find arrangements which are realizable as rectangular dissections and which satisfy the adjacency requirements;
(ii) express these arrangements as rooted simple $[r, 1]$-triangulations;
(iii) construct valid colorings for these triangulations;
(iv) derive the corresponding networks;
(v) solve the network equations with the constraints imposed by the adjacency and dimension requirements;
(vi) construct the rectangular dissections.

We may compare the procedure outlined above with those presented by Grason [14], Teague [30], and Mitchell, Steadman and Liggett [25]. Grason used the augmented dual throughout, and assigned weights to the edges representing the lengths of the corresponding walls. Teague, on the other hand, used only the network representation. It seems clear from the previous analysis that the representations suitable for dealing with the adjacencies and dimensions are the augmented dual and network representations, respectively. Mitchell, Steadman and Liggett recognized this division of the problem, but approached it in a different way by means of a catalogue of rectangular dissections.

7. Some Other Applications of Graph Theory in Architecture

The road system in a town or city can be represented by a network in which the usual network analysis of flows and capacities may be applied. However, at the initial stages in the design of such road systems, it is often important to consider the spatial disposition of the roads and the spatial layout which they define. The sets of possibilities for road networks may be considered in the same way as for floor plans, but now the edges of the plane maps represent roads. Different interpretations can be given to the various characteristics of the map, and the ornamentation operations can be modified to incorporate the details of road junctions. Rectangular dissections, in particular, may be considered as road systems laid out in a grid, in which the roads define the boundaries of rectangular blocks. It is only after such possibilities for the underlying structures have been investigated, that network analysis may be applied.

The design of a circulation system in a building is a similar problem for which a structural approach is desirable at the initial stages. In large buildings or building complexes, the circulation system consists of a series of corridors and halls which may be represented by the edges of a plane map. In most cases these plane maps are trees, or have only one or two circuits. The individual spaces have access to this core structure, and appending such individual units is among the ornamentation operations emphasized in this case. A design objective may be to minimize circulation costs within the building. An early example concerning the optimal layout of hospitals appeared in Souder *et al.* [28]. Typical graph-theoretical problems in this context are the shortest path problem and related traffic assignment problems. March and Steadman [23] and Tabor [29] also discuss the mean distance in the circulation graph as a measure of the "compactness" of the architectural plan. Recently, Doyle and Graver [10], [11] established some theoretical results on the mean distances in graphs and directed graphs, particularly in trees and rectangular grids. It might also be of interest in certain situations to interconnect a given set of locations with a circulation structure in some minimal way. This corresponds to the Steiner problem in geometry and graph theory, discussed by Chang [7] and Hanan [16]. The problem for schemes in which the locations and circulation paths lie on a rectangular grid has been discussed by Matela and O'Hare [24].

In conclusion, we mention some considerations of design problems in general. Manheim [21] represented a design problem as a hierarchical structure or a decision tree. An architectural problem is then considered as a branching tree of sub-problems, sub-sub-problems, and so on. In graph-theoretical terms, the designer needs to make use of decision tree search methods (see Christofides [8]). However, this is based on the assumption that the total problem can be partitioned into sets of sub-problems, and not all design theorists agree that this may be done without loss of integrity. In other words, is the whole merely the sum of its parts, or is it more? (see Atkin [2]). During the early 1960s, Alexander [1] suggested a method for treelike decomposition of the architectural design problem, and illustrated the approach by means of a graph representing the interactions between identifiable elements of the problem. This graph was then decomposed into subgraphs, each of which could be recognized as a sub-problem. This technique finds its parallel in the design method proposed by Kron [19] for engineering systems, which he called "diakoptics" or "tearing" (see Chapter 4). Finally these decomposition methods in architectural design were discussed by several authors in Moore [26], and briefly in Harary *et al.* [17], in a survey of the uses of k-colored graphs in design.

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