

Consider the number of r -valent labeled graphs with n vertices, where multiple edges and loops are allowed. Any such graph can be represented as follows.

First, label the vertices $1, \dots, n$, where as shown below, nr must be even. Let (l_{ij}) be a square matrix defined as follows. Let $l_{ii} = h$ if vertex i has h loops, 0 otherwise. For $i < j$, let l_{ij} be the number of edges connecting vertices i and j . For $i > j$, $l_{ij} = 0$. This is an upper-triangular matrix (including the diagonal). Now, for any i consider the sum

$$\sum_{j=1}^n (l_{ji} + l_{ij}) \quad (1)$$

This is the sum of the elements of l over row i plus the sum over column i . For any i , the sum over j of l_{ji} is the number of edges connecting vertex i to a vertex of equal or lower position. For any i , the sum over j of l_{ij} is the number of edges connecting vertex i to a vertex of equal or higher position. If there are loops at vertex i , the number of loops is counted in each sum. So the full sum is the number of edges connected with vertex i , counting the number of loops at vertex i twice, once for each end. This sum is r for all i since the graph is r -valent. Note that

$$\sum_{i=1}^n \sum_{j=1}^n (l_{ji} + l_{ij}) = 2 \sum_{i=1}^n \sum_{j=1}^n l_{ij} \quad (2)$$

That is, this is an even number. But the sum is also nr , so that only even values of nr are possible for such graphs, and at least one of n and r must be even.

A central moment of the multivariate normal distribution with exponent r for all components is $E[X_1^r \cdots X_n^r \mid \mu = 0, \Sigma]$. From Phillips (2010), even moments, in this case those with even values of nr , can be represented symbolically with the set of upper-triangular, positive integer matrices l which satisfy

$$\sum_{j=1}^n (l_{ji} + l_{ij}) = r \quad (3)$$

This set of matrices is the same as the set of matrices just defined for r -valent labeled graphs with n vertices, so for even nr , the numbers of moment representations and the numbers of such graphs are equal.