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Maximal Partial Steiner Triple Systems of Order $V \leq 11$

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1.

A *(partial) triple system* $TS(\lambda;v)$ is a pair (V,B) where V is a v-set of *elements* and B is a collection of 3-subsets of V, called *triples* such that each 2-subset of V is contained in (at most) exactly λ triples. The parameters λ and v are called the *index*, and the *order* of the triple system, respectively. When $\lambda = 1$, we have a *(partial) Steiner triple system* (*STS*).

The *leave* of a partial triple system (V,B) is a (multi)-graph (V,E) where E contains all pairs that appear less than λ times in B; the multiplicity of an edge is λ minus the number of its appearances in B. A partia[l] triple system is *maximal* if its leave is triangle-free.

In this paper we determine all maximal partial Steiner triple system[s] of order v for $v \leq 11$. We list all these systems in the Appendix, togethe[r] with their leaves and selected invariants and properties (group order chromatic index, size of minimum embedding).

A companion technical report [3] contains all proofs concernin[g] minimum embeddings, immersions and enclosings, as well as a brie[f] description of the computer algorithms used to generate the systems.

2.

A maximal partial Steiner triple system of order v is denoted $mpt(v$[)]. An $mpt(v)$ is *maximum (minimum)* if its leave has the fewest (most[)] edges of any $mpt(v)$; such a system is denoted $Mmpt(v)$ ($mmpt(v)$).

The number of triples in an $Mmpt(v)$, and, in fact, the leaves [of] $Mmpt(v)$'s, as well-known (cf., eg., [4] or [2]); the leaves are unique up t[o] an isomorphism. The number of triples in an $mmpt(v)$ was determined b[y] Novák [6]; unlike maximum systems, the leaves of minimum systems a[re] not necessarily unique. The possible number of triples in any $mpt(v)$ (i.[e.] the spectrum for $mpt(v)$'s) was then determined almost completely b[y] Severn [8]. Roughly speaking, the spectrum covers, with a few exception[s] the interval between the least and largest possible number of triples in [a]

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mpt(v).

As far as we can tell, the only enumeration results for maximal partial triple systems (apart from those for *STS*s, of course) are due to Novak [5] who produced all *mpt*(v)'s for $v \leq 9$ (with one omission for $v = 9$); some examples of *mpt*(11)'s are given in [7]. We extend here the catalogue of *mpt*(v)'s up to $v \leq 11$. The number of *mpt*'s is summarized in the following table:

Order v	3	4	5	6	7	8	9	10	11
Number of *mpt*(v)	1	1	1	2	2	4	10	47	472
Number of leaves	1	1	1	2	2	4	9	22	85

We list all these systems in the Appendix, together with their leaves. We also present, for each *mpt*, the order of its automorphism group, its chromatic index, and the size of its minimum embedding. We have also obtained for each *mpt*(v) the minimum immersion constant and the minimum enclosing constant but we do not list these with the individual systems as these are almost always the same for the systems of given order but rather comment on these in the next section. One "obvious" invariant that we do not list is the chromatic number; in our range, this invariant does not yield significant information as its value is always 2 or 3, with the former occurring only infrequently. The following are the necessary definitions.

A *k-block-colouring* of a partial *TS* is an assignment of k colours to its triples such that any two intersecting triples are assigned distinct colours. The *chromatic index* of a partial *TS* is the least k for which the *PTS* has a k-block-colouring.

A partial *STS*(V,B) is said to be *embedded* in an *STS*(W,C) if $V \subset W$ and $B \subset C$. The containing system (W,C) is an *embedding of the PTS*(V,B). *The number* $|W|$ *is the size of minimum embedding* for (V,B) if for any embedding of (V,B) into (U,A), $|W| \leq |U|$. A partial *STS*(V,B) is said to be *immersed* in a $TS(\lambda;|V|)$ (V,D) if $B \subset D$; the containing system (V,D) is an *immersion* of (V,B).

Similarly, a partial *STS*(V,B) is said to be *enclosed* in a $TS(\lambda,w)$ (W,E) if $V \subset W$ and $B \subset E$; the containing system (W,E) is an *enclosing* of (V,B). For a partial *STS* $S = (V,B)$, define

$$\upsilon(S) = \min\{\lambda-1\colon \text{ there exists an immersion of } S \text{ into } TS(\lambda;|V|)\}$$

$$\mu(S) = \min\{\lambda;-1+w=v\colon \text{ there exists an enclosing of } S \text{ into } TS(\lambda;w)\}.$$

The numbers $\upsilon(S)$ and $\mu(S)$ are termed the *minimum immersion constant*, and the *minimum enclosing constant*, respectively. Finally, let $\upsilon(v) = \max\upsilon(S)$, $\mu(v) = \max\mu(S)$ where the maxima are taken over all *mpt*(v)'s (cf. [2]).

3.

The values of $\upsilon(v)$ and $\mu(v)$ for $v \leq 11$ are given in the followint table:

v	3	4	5	6	7	8	9	10	11
$\upsilon(v)$	0	1	2	1	1	5	1	1	2
$\mu(v)$	0	1	2	1	1	2	1	1	2

In fact, $\upsilon(S) = \upsilon(v)$ and $\mu(S) = \mu(v)$ for each *mpt*(v) S with $v \leq 11$, with the obvious exceptions of the *STS*s of order 7 and 9; for the *STS*s, trivially $\upsilon(S) = \mu(S) = 0$.

Moreover, each *mpt*(11) has an immersion into a $TS(3;11)$ as well as an enclosing in a $TS(2;12)$!

It follows from this table that both Conjecture I and Conjecture II from [2] hold for $v \leq 11$.

Information about the automorphism group orders is summarized in the next table where the symbol G represents the group order and # the number of systems with this group order.

Automorphism group orders of the mpt's

v=7

b=5

G	12
#	1

b=7

G	168
#	1

v=8

b=7

G	168	8	4
#	1	1	1

b=8

G	48
#	1

v=9

b=8

G	48	24	8	4
#	1	1	1	1

b=9

G	9	2	1
#	1	1	1

b=10

G	12	6
#	1	1

b=12

G	432
#	1

v=10

b=8

G	1008
#	1

b=10

G	120	24	10	6	4	2
#	1	3	1	1	2	2

b=11

G	3	2	1
#	2	4	9

b=12

G	432	16	12	6	4	3	2	1
#	1	1	2	1	5	1	4	4

b=13

G	6	3
#	1	1

v=11

b=11

G	24	11	6
#	1	1	1

b=12

G	6	1
#	1	8

b=13

G	192	96	24	16	12	8	6	4	3	2	1
#	1	1	3	1	1	3	1	11	3	33	74

b=14

G	3	2	1
#	2	14	193

b=15

G	20	12	10	6	4	3	2	1
#	1	1	1	1	7	1	33	65

b=16

G	2	1
#	4	3

b=17

G	8	1
#	1	1

In this connection it is interesting to observe that for each of the orders 7, 8, 9, 10 and 11 there exists, up to an isomorphism, a unique cyclic maximal partial Steiner triple system. Information about chromatic index is summarized below; here c represents the chromatic index and # the number of systems with this chromatic index. The results may seem bizarre at first glance but whenever a large chromatic index occurs (=7) it is due "essentially" to the fact that the system in question contains an *STS* of order 7.

Chromatic indices of the mpt's

v	b	c	#
v=7	b=5	4	1
	b=7	7	1
v=8	b=7	5 6 7	1 1 1
	b=8	4	1
v=9	b=8	5 6 7	1 1 1
	b=9	5 6	1 1
	b=10	5	1
	b=12	4	1
v=10	b=8	7	1
	b=10	5 6 7	5 3 2
	b=11	5 6	9 6
	b=12	4 5 6	4 10 5
	b=13	5	2
v=11	b=11	6 7	1 2
	b=12	5 6	4 5
	b=13	5 6 7	71 46 15
	b=14	5 6	143 66
	b=15	5 6	33 77
	b=16	6	7
	b=17	7	2

The question whether it is feasible to extend this catalogue to include all maximal partial triple systems of order 12 is hard to answer at present. There are probably several thousands of these systems. The minimum and maximum number of triples in a maximal partial *STS* of order 12 is 13 and 20, respectively. The number of *Mmpt*(12)'s (i.e. those with 20 triples) is 5; this follows easily by observing that the two well-known *STS*s of order 13 have one and four point-orbits, respectively. But the bulk of the *mpt*(12)'s will likely be given by those with a number of triples "halfway" between the minimum and maximum possible.

References.

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Appendix.

A listing of all maximal partial Steiner triple systems of order v with $6 \leq v \leq 11$ follows. For each design, an explicit listing of its blocks is followed by the name of its leave, the order of its automorphism group, its chromatic index, and the size of its minimum embedding.

The diagrams of all graphs that appear as leaves of *mpt*'s with $8 \leq v \leq 11$ are given on separate pages. For $v = 6,7$ the leaves are not shown: A_1 is $K_{3,3}$, A_2 is $3K_2$, B_1 is the hexagon and B_2 is the null graph.

number of elements = 6, number of blocks = 2

```
0 03
0 14       A1  72 1  9
1 25
```

number of elements = 6, number of blocks = 4

```
0 0012
0 1334     A2  24 4  7
1 2455
```

number of elements = 7, number of blocks = 5

```
0 00124
0 13335    B1  12 4  9
1 24566
```

number of elements = 7, number of blocks = 7

```
0 0012120
0 1333445  B2 168 7  7
1 2456656
```

number of elements = 8, number of blocks = 7

```
0 0012120                  0 0012120
0 1333445  C1 168 7 15     0 1333445     C2   8 6 15
1 2456656                  2 2456657

0 0012120
0 1333445  C3   4 5 15
3 2456677
```

number of elements = 8, number of blocks = 8

```
0 00121240
0 13334556  C4  48 4  9
1 24567787
```

number of elements = 9, number of blocks = 8

```
0 00121200                 0 00121206
0 13334457  D1  48 7 15    0 13334457    D2   8 6 15
1 24566568                 2 24566578

0 00121206                 0 00121205
0 13334457  D3   4 5 13    0 13334467    D4  24 5 13
3 24566788                 4 24566788
```

number of elements = 9, number of blocks = 9

```
0 001212001                0 001212026
0 133344567  D5   2 6 19   0 133344557   D6   1 5 19
1 245665788                2 245667788

0 001212450
0 133345667  D7   9 5 19
3 245678878
```

number of elements = 9, number of blocks = 10

```
0 0012120201               0 0012120203
0 1333445567  D8  12 5 19  0 1333445567  D8   6 5 19
1 2456677888               2 2456677888
```

number of elements = 9, number of blocks = 12


```
0 001212024013
0 133344555667  D9 432 4  9
1 245678876788
```

number of elements = 10, number of blocks = 8

```
0 00121207
0 13334458  E1 1008 7 19
1 24566569
```

number of elements = 10, number of blocks = 10

```
0 0012102146                0 0012102346
0 1333455778  E2   4 5 15   0 1333455778   E2   2 5 15
1 2456678899                2 2456678899

0 0012120012                0 0012120013
0 1333445778  E3  24 7 15   0 1333445778   E3   6 7 15
3 2456656899                4 2456656899

0 0012120057                0 0012120126
0 1333445668  E3  24 6 15   0 1333445778   E4   2 6 15
5 2456657899                6 2456657899

0 0012120146                0 0012125064
0 1333445778  E4   4 6 15   0 1333456778   E5  10 5 13
7 2456657899                8 2456789989

0 0012445012                0 0012013245
0 1333566778  E6  24 5 15   1 1334666778   E6 120 5 15
9 2456789899                0 2455789899
```

number of elements = 10, number of blocks = 11

```
0 00121024146               0 00121200125
0 13334555778  E7   2 5 13  0 13334456778  E8   1 6 19
1 24566789989               2 24566578899

0 00121200135               0 00121200145
0 13334456778  E8   1 6 19  0 13334456778  E8   1 6 19
3 24566578899               4 24566578899

0 00121200165               0 00121202407
0 13334456778  E9   2 6 19  0 13334455568  E8   1 5 19
5 24566578899               6 24566778989

0 00121200152               0 00121200153
0 13334456778  E10  3 5 13  0 13334456778  E11  1 5 13
7 24566789899               8 24566789899

0 00121200154               0 00121202061
0 13334456778  E10  3 5 13  1 13334455778  E7   2 6 13
9 24566789899               0 24566789989

0 00121202063               0 00121202064
1 13334455778  E7   1 5 13  1 13334455778  E12  1 5 13
1 24566789989               2 24566789989

0 00121202163               0 00121202164
1 13334455778  E7   1 5 13  1 13334455778  E7   1 5 13
3 24566789899               4 24566789899

0 00121205061
1 13334456778  E13  2 6 19
5 24566789989
```

number of elements = 10, number of blocks = 12

```
0 001212005123               0 001212005124
0 133344566778  E14  2 6 19  0 133344566778  E14  3 6 19
1 245665789899               2 245665789899

0 001212005143               0 001212024013
0 133344566778  E14  6 6 19  0 133344555677  E15  4 5 19
3 245665789899               4 245667789889

0 001212024013               0 001212024013
0 133344555677  E15  2 5 19  0 133344555678  E16  1 5 19
5 245667789898               6 245667789899
```