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PAIR-COVERINGS WITH RESTRICTED LARGEST BLOCK LENGTH

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1. *Introduction.*

Erdös and de Bruijn introduced the problem of determining $g(1,2;v)$, the minimal number of incomplete blocks made up of elements from a v-set in such a way that every pair occurs exactly once in the blocks selected. They showed [1] that $g(1,2;v) = v$ and that this minimum is always given by a near-pencil: one block of length $v-1$ plus $v-1$ pairs. Exceptionally, the minimum is also attained if $v = k^2-k+1$ and a geometry of k points per line exists; then the geometry covers all pairs with v lines comprising k points each.

Woodall [6] considered $g(1,\mu;v)$; this is the cardinality of the minimal family of sets chosen from a v-set in such a way that every μ-set occurs exactly once. Woodall showed that

$$g \geq 1 + (v-k)\binom{k}{\mu-1}\left(1 - \frac{v-k-1}{2(k-\mu+2)}\right) \tag{1.1}$$

where k is the size of a block in the covering set (henceforth, we shall always use k as the size of the largest block in the covering set). Stanton and Kalbfleisch [4] showed that

$$g \geq 1 + \frac{k-\mu+2}{v-\mu+1}\binom{k}{\mu-1}(v-k) \; . \tag{1.2}$$

And it is trivial, by a counting argument, to obtain the bound

$$g \geq \binom{v}{\mu}\Big/\binom{k}{\mu} \; . \tag{1.3}$$

If we specialize these three bounds to the case $\mu = 2$, we have the lower bounds

$$W = 1 + \frac{v-k}{2}(3k-v+1) \; , \tag{1.4}$$

$$SK = 1 + \frac{k^2(v-k)}{v-1} \; , \tag{1.5}$$

$$C = v(v-1)/k(k-1) \; . \tag{1.6}$$

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It was shown (see [2], or, for more detail, [3]) that the SK bound (1.5) easily produces the Erdős-de Bruijn Theorem.

In this paper we introduce the numbers

$$g^{(k)}(1,2;v) = g^{(k)}(v)$$

as the cardinality of the minimal family of sets that covers all pairs, given that the elements are from a v-set and that the size of the longest block in a covering family is k. When the argument v is obvious, we simply write $g^{(k)}$.

2. *An Example.*

Suppose that we take $v = 13$; then we can construct the following table. The values $g^{(k)}(1,2;13)$ for $k > 6$ will be justified in the next section. Note that

$$W = 1 + \frac{3}{2}(13-k)(k-4),$$

$$SK = 1 + \frac{k^2(13-k)}{12},$$

$$C = 156/k(k-1).$$

k	$g^{(k)}$	W	SK	C
13	1	1	1	1
12	13	13	13	2
11	22	22	22	2
10	28	28	26	2
9	31	31	28	3
8	31	31	28	3
7	28	28	26	4
6	24	22	22	6
5	19	13	18	8
4	13	1	13	13
3	26	neg	9	26
2	78	neg	5	78

For $k = 2,3,4$, the values follow from using the set of all pairs; from using the triple system on 13 elements; and from using the projective geometry on 13 elements.

3. *The Construction of a System for Large* k.

Suppose that $k = v-2\alpha$; then the Woodall bound is

$$W = 1 + \alpha(2v-6\alpha+1) .$$

Now take a complete 1-factorization of the 2α points not contained in the block of length k. Form triples by associating all pairs in any 1-factor with the same point in the block of length k (this can be done so long as the number of 1-factors, which is $2\alpha-1$, is not greater than $v-2\alpha$). Use all pairs not contained in the block of length k or in the triples. Then the total number of blocks is

$$1 + (2\alpha-1)\alpha + \binom{v}{2} - \binom{v-2\alpha}{2} - \alpha(2\alpha-1)3$$

$$= 1 + \alpha(2v-6\alpha+1) = W.$$

The condition $2\alpha-1 \le v-2\alpha$ simplifies to

$$\alpha \le (v+1)/4.$$

Thus $k \ge v-(v+1)/2 = (v-1)/2$, and we have

THEOREM 3.1. *If* $v-k$ *is even, then the Woodall bound gives* $g^{(k)}$ *for* $k \ge (v-1)/2$.

For $v-k$ odd, we need a different factorization of the pairs on the $v-k = 2\alpha+1$ points. The easiest way to get the complete 1-factoriza- of an even number of points (take 8 as an example) is to place 1 at the centre of a circle formed by the other 7 points, as shown in Figure 1.

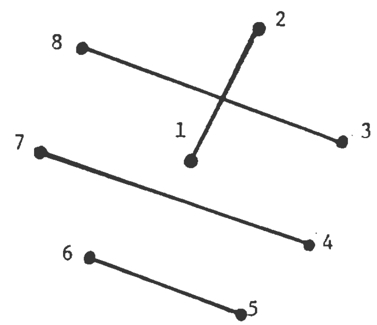
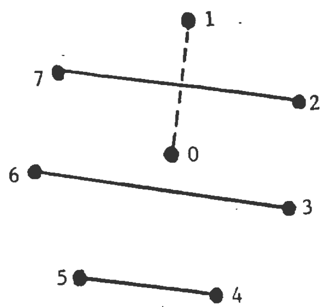


Figure 1: 1-Factorization of K_8.

The first 1-factor is found by taking (1,2) and the three perpendicular chords, as shown. The other 1-factors are found by rotating this 1-factor about 1.

Similarly, if there is an odd number of points, say 7, we can place them on the circumference of a circle with centre 0.



The first "1-factor" is found by taking "1", joining 0 to 1, and taking the chords perpendicular to (0,1). By rotation, we get 7 generalized 1-factors.

Now we make a construction analogous to that in Theorem 3.1. We form triples by taking all pairs in a 1-factor and adjoining the same point from the k-block. This is possible so long as

$$2\alpha+1 \le v-(2\alpha+1) ,$$

that is, so long as

$$\alpha \le (v-2)/4 ,$$

or

$$k \ge v-(v-2)/2-1 = v/2 .$$

The number of pairs needed to cover all pairs is now found as

$$\binom{v}{2} - \binom{v-2\alpha-1}{2} - 3\alpha(2\alpha+1)$$
$$= 2v\alpha-8\alpha^2-6\alpha+v-1.$$

So the total number of blocks is this number increased by $1+\alpha(2\alpha+1)$, that is,

$$2v\alpha-6\alpha^2+v-5\alpha .$$

But we easily calculate that

$$W = 1 + \frac{2\alpha+1}{2}(3v-6\alpha-3-v+1)$$
$$= 1 + (2\alpha+1)(v-3\alpha-1)$$
$$= 2\alpha v-6\alpha^2+v-5\alpha.$$

We thus have

THEOREM 3.2. *If* $v-k$ *is odd, then the Woodall bound gives* $g^{(k)}$ for $k \ge v/2$.

These two theorems are easily merged into

THEOREM 3.3. *If* $v \equiv 1 \pmod 4$, *the Woodall bound holds for* $k > (v-1)/$ *otherwise, the Woodall bound holds for all* $k \ge (v-1)/2$. *Thus, for* k *in these ranges, we have* $g^{(k)} = 1 + (v-k)(3k-v+1)/2$.

Actually, we can go a bit farther. If we use (2.1) and (2.2) fro [3], it follows that, with $t = 2$, the Woodall bound W is only attain if

$$\sum_{A(0)} \binom{k_i}{2} = 0 ,$$

where $A(0)$ is the set of blocks disjoint to the block of length k, is the set of lengths for these blocks. Thus, we have the result that a other blocks meet the longest block. Furthermore, it is also required t

$$\sum_{A(1)} \binom{k_i-2}{2} = 0 ,$$

and this shows that the blocks meeting the longest block in 1 point (all others) have cardinalities 2 or 3. Thus we have

THEOREM 3.4. *The only configurations producing the bound* W *are those using pairs and triples, as described earlier in this section.*

4. *The Values* $k=2$ *and* 3.

It is trivial to note that

$$g^{(2)} = \binom{v}{2} .$$

Also, it is clear that $g^{(3)}$ is obtained by taking as many triples as possible; now this number (see, for example, [5]) is

$$D(2,3,v) = \left[\frac{v}{3}\left[\frac{v-1}{2}\right]\right] - \delta_{\alpha 5}$$

where $[x]$ denotes the greatest integer not exceeding x, and α is the congruence class of v, modulo 5. It follows that we can state

THEOREM 4.1. *The value of* $g^{(2)}$ *is* $\binom{v}{2}$, *and the value of* $g^{(3)}$ *is*

$$\binom{v}{2} - 2\left[\frac{v}{3}\left[\frac{v-1}{2}\right]\right] + 2\delta_{\alpha 5} .$$

It is useful to record $g^{(3)}$ according to the form of v. We give two alternative forms.

v	$g^{(3)}$	$g^{(3)}$
6t	$6t^2+t$	$v(v+1)/6$
6t+1	$6t^2+t$	$v(v-1)/6$
6t+2	$6t^2+5t+1$	$v(v+1)/6$
6t+3	$6t^2+5t+1$	$v(v-1)/6$
6t+4	$6t^2+9t+4$	$(v^2+v+4)/6$
6t+5	$6t^2+9t+6$	$(v^2-v+16)/6$

Using the results proved so far, we can fill in the following table.

k \ v	2	3	4	5	6	7	8	9	10	11	12
2	1	3	6	10	15	21	28	36	45	55	66
3		1	4	6	7	7	12	12	19	21	26
4			1	5	8	10	11	12*	12*	13*	13*
5				1	6	10	13	15	16	16	18*
6					1	7	12	16	19	21	22
7						1	8	14	19	23	26
8							1	9	16	22	27
9								1	10	18	25
10									1	11	20
11										1	12
12											1

The starred values do not follow from our theorems; rather we need some easy Lemmata.

LEMMA 4.2. $g^{(4)}(9) = 12$.

Proof. The SK bound is 11; also it is clear that

1234	189	368	269	459	35
1567	258	478	379	27	46

provides a cover in 12 blocks.

If a pairs, b triples, c quadruples, provide a cover in 11 blocks, then

$$a+b+c = 11, \; a+3b+6c = 36.$$

It follows that $2b+5c = 25$, and we have one of 3 cases:

(1) $c = 5$, $b = 0$, $a = 6$;

(2) $c = 3$, $b = 5$, $a = 3$;

(3) $c = 1$, $b = 10$, $a = 0$.

Case (1) is impossible, since $D(2,4,9) = 3$. Case (2) is not possible sin we get 1234, 1567, 4789; then we can not have 5 triples. And Case (3) i impossible since the quadruple 1234 leaves 5 symbols to go with 1, and hence the use of triples only is impossible.

Indeed, Lemma 4.2 generalizes trivially to the result.

LEMMA 4.3. *If* $v = t^2$, then $g^{(k)}(v) \geq t^2+t \, (t \geq 3)$.

Proof. It is clear that

$$g^{(t^2-2)}(v) = 2t^2-4 > t^2+t$$

by using Theorem 3.3.

Also, the counting bound shows that

$$g^{(t)}(v) \geq \frac{t^2(t^2-1)}{t(t-1)} = t^2+t.$$

Finally, the SK bound gives

$$g^{(t+1)}(v) \geq 1 + \frac{(t+1)^2(t^2-t-1)}{t^2-1},$$

that is,

$$g^{(t+1)}(v) \geq t^2+t-\frac{2}{t-1} .$$

This shows that $g^{(t+1)}(v) \geq t^2+t$ for $t > 3$, and the result for $t =$ [illegible] follows from Lemma 4.2.

The general result then follows from the shape of the bounding curve

$$SK = 1 + \frac{k^2(v-k)}{v-1} = 1 + \frac{k^2(t^2-k)}{t^2-1}$$

between $k = t+1$ and $k = t^2-2$.

LEMMA 4.4. $g^{(4)}(10) = 12$.

Proof. The SK bound is 12, and the cover

1234	258	26T	279
1567	369	378	468
189T	47T	459	35T

is trivially obtained.

LEMMA 4.5. $g^{(4)}(11) = 13$.

Proof. Again, the SK bound is 13. Simply, take an affine geometry on 9 points, adjoin T and E to two resolution classes, and add the pair {T,E}.

LEMMA 4.6. $g^{(4)}(12) = 13$.

Proof. The SK bound of 13 is achieved by deleting a single point from the 13-point geometry.

For $g^{(5)}(12)$, matters are slightly more complicated; it is easy to get the SK bound of 17 and the construction

12345	T62	E63	V64
16789	T73	E74	V75
ITEV	T84	E85	V82
	T95	E92	V93

along with pairs 65, 72, 83, 94, shows that $g^{(5)}(12) \leq 19$.

Now blocks 12345, 6789T, imply at least 2 + [illegible] = 27 blocks. Blocks 12345, 16789, imply at least 3 + 16 = 19 blocks. So there can be only one block of length 5 if $g^{(5)}(12) < 19$. Let this block be B = {89TEV}. Then use a pairs, b triples, c quadruples, and we have

$$a+b+c = 16 + \delta \quad (\delta = 0 \text{ or } 1),$$
$$a+3b+6c = 56.$$

Then $2b+5c = 40 - \delta$, and we have cases:

(1) $\delta = 0$; $c = 6$, $b = 5$, $a = 5$
(2) $\delta = 0$; $c = 4$, $b = 10$, $a = 2$
(3) $\delta = 1$; $c = 7$, $b = 2$, $a = 8$
(4) $\delta = 1$; $c = 5$, $b = 7$, $a = 5$
(5) $\delta = 1$; $c = 3$, $b = 12$, $a = 2$.

Now no quadruple is disjoint to B, or we would have at least 2 + 20 = 22 blocks. If there is at most one quadruple through any point of B, then $c \leq 5$; also, if 2 quadruples pass through one point in B, we find that only 3 more are possible. This rules out Cases (1) and (3).

In Case (4), our 5 quadruples use up 5 triples from A = {1,2,... So we can only get triples by using an element from B with a pair from A; since only 21 - 15 = 6 pairs are available, we can not meet the requirement b = 7.

In Case (2), we only need 4 quadruples. This leaves 9 pairs free in A; but, even using all of them, we can not get 10 triples. Hence, we need only consider the case

$$a = 2, \quad b = 12, \quad c = 3; \quad \delta = 1.$$

This can only occur if 3 triples from A are used for quadruples and the other 12 pairs from A are used to form triples. Then each point in B must occur with 3 or 1 points from A; hence the distribution of lines through the points of B is 3(1 quadruple, 2 triples), 2(1 pair, 3 triple
We may thus form the blocks:

89TEV, 8123,
845, 867;
9146, 925, 937.

If we now take T157, T24, T36, then we are forced to have E1 and V1. Triples E26, E35, E47 are available; so are V27, V34, V56. Thus we have achieved a construction and established

LEMMA 4.7. $g^5(12) = 18$.

5. *The Case* $v = 13$.

It will be useful to give a slight strengthening of the SK bound before we complete the Table in Section 2.

From the derivation in [3], we see that the SK bound comes from using a positive variance and omitting the set A_0. Thus we have

LEMMA 5.1. *If the SK bound is an integer and if it gives the exact value of* $g^{(k)}$, *then all other blocks meet the block of length* k *and all of these other blocks have the same length* t.

Indeed, it follows at once that these other blocks form a BIBD with $1+k(t-2)$ varieties, block size $t-1$, $\lambda = 1$, and this BIBD is resolvable into k resolution classes. It further follows that $t-1$ divides $k-1$.

There are 3 obvious cases in which the bound is exact. If $t = 2$, then $v = k+1$ and we have a near-pencil. If $t = k$, then $v = k^2-k+1$ and we have a projective geometry (in appropriate cases). If $t = 3$, and $v = 4m+3$, then $k = (v-1)/2 = 2m+1$, and we have one of the cases covered earlier.

However, if $v = 4m+1$, $k = (v-1)/2 = 2m$, we have $SK = 1+(v^2-1)/8$. This is an integer, but the $2m+1$ points not in the long blocks can not be partitioned into pairs to form triples. Thus Lemma 5.1 gives us

LEMMA 5.2. *If* $v = 4m+1$, $k = 2m$, *then the number of blocks strictly exceeds the bound* $1 + (v^2-1)/8$.

Now consider $g^{(5)}(13)$. The SK bound gives $g^{(5)}(13) \geq 18$. An easy construction

∞ 1234,	∞ 5678,	∞ 9TEV	
159	25T	35E	45V
16T	26E	36V	469
17E	27V	379	47T
18V	289	38T	48E

shows that $g^{(5)}(13) \leq 19$.

If $g^{(5)}(13) = 18$, we first note that any other block must meet th initial base block $B = \{\infty\ 1234\}$. For using the exact relation (2.5) from [3], we find that the number of blocks is at least

$$1 + \frac{200}{12 - \frac{1}{4}(a_0+3b_0+6c_0+10d_0)}$$

where there are a_0 blocks of length 2 disjoint from the base block, b_0 blocks of length 3, etc. (of course, it is clear that $c_0 = d_0 = 0$). Even $a_0 = 1$, $b_0 = 0$, gives a bound of 19. So we find that all blocks meet the base block.

If there is a second block of length 5, we can immediately form at least $3 + 16 = 19$ blocks. So take a pairs, b triples, c quadruples with

$$a + b + c = 17$$
$$a + 3b + 6c = 68.$$

Then $2b + 5c = 51$, whence we find:

(1) $b = 3$, $c = 9$, $a = 5$

(2) $b = 8$, $c = 7$, $a = 2$.

There can be at most 2 quadruples through any point on B. From this we find there is no distribution of pairs and triples to points of B that works in Case (1) or Case (2). Hence we have

LEMMA 5.3. $g^{(5)}(13) = 19$.

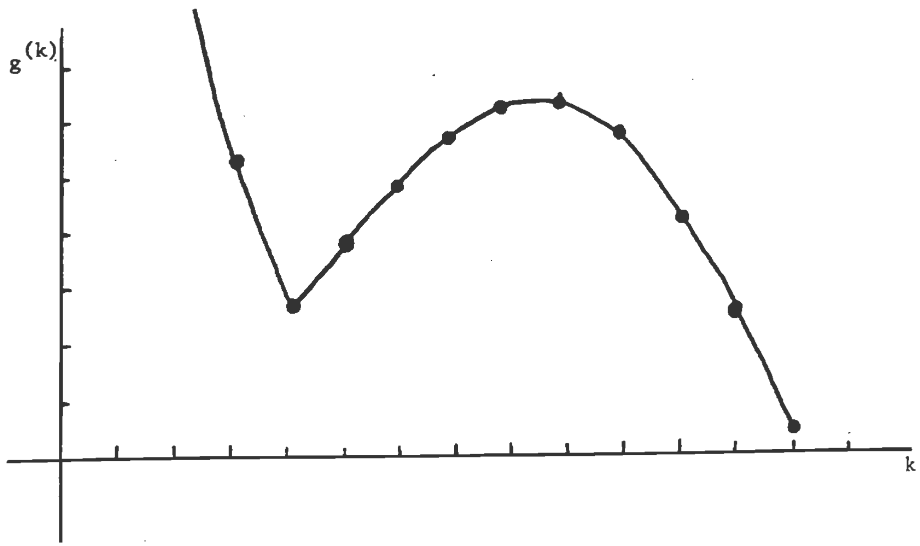
We now move to the case $k = 6$ and use Lemma 5.2 to give the bound o Actually $g^{(6)}(13) = 24$; this is a special case of the result

$$g^{(2a)}(4a+1) = 2a^2 + a + 1 + \lceil a/2 \rceil$$

which will be shown in a forthcoming paper on graph factorization.

6. *Values of* $g^{(k)}(v)$ *which are near* v.

Let us return to Section 2 and plot $g^{(k)}(v)$.



Ignoring the case when $k = v$, which is trivial, we note that $g^{(k)}$ is usually much greater than v. Indeed we can prove

THEOREM 6.1. *If* $v = t^2+t+1+e$, where $0 \le e < 2t+2$, and if $g^{(k)}(v) > v$, then (with a few small exceptions)

$$g^{(k)}(v) \ge t^2+3t+1$$

unless $k = t+1$.

Proof. From the shape of the graph of $g^{(k)}(v)$, it is clear that we need only consider the cases $k = v-2$, $k = t$, $k = t+1$, $k = t+2$.

For $k = v-2$, we have

$$g^{(k)} = 2v-4 = 2t^2+2t-2+2e = t^2+3t+1 + (t^2-t-3+2e) .$$

Now $g^{(k)} > t^2+3t+1$ for $t \ge 3$. The only exception is for $t = 2$; there $g^{(5)}(7) = 10$.

For $k = t+1$, we use SK and have

$$g^{(k)}(v) \ge 1 + \frac{t^2+e}{t^2+t+e}(t+1)^2$$

$$= 1 + \frac{t^4+2t^3+(e+1)t^2+2et+e}{t^2+t+e}$$

$$= 1 + t^2 + t + \frac{e(t+1)}{t^2+t+e} .$$

For $k = t+2$, we have

$$g^{(k)}(v) \ge 1 + \frac{(t+2)^2(t^2+e-1)}{t^2+t+e}$$

$$= 1+t^2+3t+ \frac{(t+4)(e-4)}{t^2+t+e} .$$

Clearly, the only exceptions that can occur are for $e=0(t=2,3,4,5)$, $e=1(t=2,3,4)$, $e = 2(t=2,3)$.

For $k = t$, we use the C bound and have

$$g^{(k)}(v) \ge \frac{(t^2+t+1+e)(t^2+t+e)}{t(t-1)}$$

$$= t^2+3t+2e+5 + \frac{(4e+6)t+(e^2+e)}{t^2-t} .$$

Thus we have $g^{(k)}(v) \ge t^2+3t+1$ unless t and e have certain small values, so long as $k \ne t+1$.

COROLLARY. *If* $v = t^2+t+1+e$, $k = t+1$, $g^{(k)}(v) < t^2+2t+1$, *then the number of blocks of length* $t+1$ *is at least* $(t+1)(t+2)/2$.

Proof. The worst case is when all other blocks have length t. With obvi[ous] meanings for x and y, we have

$$x+y = t^2+2t+1-a,$$

$$(t+1)tx+(t-1)ty = (t^2+t+1+e)(t^2+t+e) .$$

Then $$2tx = (t^2+t+1+e)(t^2+t+e) - (t^2-t)(t^2+2t+1-a)$$

$$= t^3+(3+2e+a)t^2 + (2+2e-a)t + e^2+e .$$

s $x \geq \frac{1}{2}(t^2+3t+2)$, even for $a = e = 0$.

Note added: Paul Erdös has drawn our attention to an interesting question concerning non-minimality. Suppose that one has a covering family, not necessarily minimal, and suppose that this family has more than v blocks. How close to v can the number of blocks be? The results of this section show that the only case needing to be considered is the case $k = t+1$.

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THE NON-EXISTENCE OF A (2,4)-FRAME

D.R. Stinson

Abstract

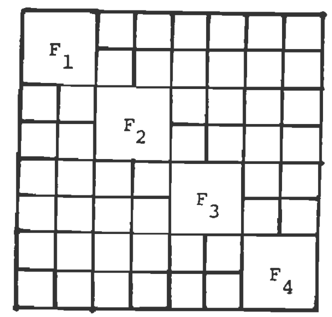
It is shown that a (2,4)-frame does not exist.

1. *Introduction.*

A (2,4)-frame, if it were to exist, would be an eight-by-eight array F of cells with the following properties:

(1) A cell either is empty or contains an unordered pair of elements chosen from the set $S = \{1,2,3,4\} \times \{1,2\}$.

(2) There are four empty two-by-two blocks F_1, F_2, F_3, F_4, down the diagonal of F:



(3) A row or column which meets F_i contains precisely the elements of $S \backslash (\{i\} \times \{1,2\})$.

(4) The unordered pairs occurring in F are precisely those $\{(k,m),(\ell,n)\}$ where $k \neq \ell$.

For any ordered pair of positive integers (t,u), a (t,u)-frame is defined analogously. Frames have been of considerable use in the construction of Howell designs and Room squares (see [1] and [4]). The following existence results have been shown.

THEOREM 1.1 (Dinitz and Stinson [2])

(1) If $u \geq 6$, then a (t,u)-frame exists if and only if $t(u-1)$ is even,

(2) If $\gcd(t,210) \neq 1$, then a $(t,5)$-frame exists,

(3) If t is a multiple of four, then a $(t,4)$-frame exists, whereas if t is odd, then no $(t,4)$-frame exists,

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