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Two orthogonal (8,4)-frame starters

21,22	25,27	14,17	1,6	5,11	23,30
14,15	19,21	27,30	2,7	3,9	26,1
10,19	31,9	2,13	26,7	15,29	3,18
29,6	13,23	11,22	5,18	17,31	10,25

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ON THE MAXIMUM NUMBER OF CYCLES IN A GRAPH+

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Abstract

Let G be a graph with p vertices and $q = n+p-1$ edges. Let $\Psi(n)$ be the maximum number of cycles such graphs can have. We show that $2^{n-1} < \Psi(n) < 2^n$ and conjecture that the lower bound is asymptotically correct. We also show that for each $n \geq 3$ there exists a cubic graph G_n with $2n-2$ vertices such that G_n has $\Psi(n)$ cycles. The upper bound for $\Psi(n)$ could be substantially improved, if, as we conjecture, every cubic graph G other that K_4 and $K_{3,3}$ contains an edge in at most half of the cycles of G.

Let G be a simple, connected graph with p vertices and q edges. Our investigation began with the following question. What (topological) structure for G would maximize the total number of cycles of any length? In this note we present some of the many interesting (unanswered) questions and conjectures that have been developed.

Let $\Psi(G)$ denote the total number of cycles in G. For example, $\Psi(K_4-x) = 3$; $\Psi(K_{3,3}) = 15$ with nine 4-cycles and six 6-cycles; and if P is the Petersen graph, then $\Psi(P) = 57$ from twelve 5-cycles, ten 6-cycles, fifteen 8-cycles and twenty 9-cycles. If edge $e = (u,v)$ is an edge in G, let $\Psi(G;e)$ or $\Psi(G;u,v)$ denote the total number of cycles in G that contain edge e. In $K_{3,3}$ there are $4 \cdot 9/6 \cdot 6 = 72$ *inclusions* of edges in cycles, and, by symmetry, each of the 9 edges is on $72/9 = 8$ cycles. That is, $\Psi(K_{3,3};e) = 8$ for each edge $e \in E(K_{3,3})$.

If $q = p-1$, then G is a tree and $\Psi(G) = 0$. If $q = p$, then G is unicyclic and $\Psi(G) = 1$. The structures maximizing Ψ for $q = p+1$, $p+2$ and $p+3$ are presented in Figure 1.

Let $n = q-p+1$. It is well known that the cycle space of G has dimension n, and thus a simple upper bound for the number of cycles in G is $2^n - 1$. We are interested in maximizing Ψ as a function of n and define $\Psi(n)$ to be the maximum possible value of $\Psi(G)$ for all those graphs G with $q-p+1 = n$.

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In Maurer [3] graphs G with 2^n-1 cycles are called cycle-complete. Such graphs are characterized in Maurer [3] and in Mateti and Deo [2] as follows. Graph G can be *reduced* by deleting any vertex of degree one or, if G contains a vertex u of degree two which is adjacent to vertices v and w with $(v,w)\notin E(G)$, by replacing u by the edge (v,w).

THEOREM 1. *Graph G is cycle-complete if and only if G is reducible to K_1, K_3, K_4-e, K_4 or $K_{3,3}$.*

It is our belief that $\Psi(n)$ essentially drops from the upper bound of 2^n-1 to one-half of this value. That is, $\Psi(n)$ is approximately 2^{n-1}.

THEOREM 2. *For every $n \geq 3$ there exists a cubic graph G_n with $q-p+1 = n$ and $\Psi(n) = \Psi(G_n)$.*

Proof. Assume H_n is a graph with $q-p+1 = n$ and $\Psi(n) = \Psi(H_n)$. Suppose v is a vertex in H adjacent to vertices $u_1, u_2, \ldots, u_d$. If $d \geq 4$ one can vertex split v as follows. Let H_n' be the graph obtained from H_n b deleting vertex v and the d edges incident with it, adding new vertices v_1 and v_2, and adding edges $(v_1,v_2),(v_2,u_3),\ldots,(v_2,u_d)$, (v_1,u_1) and (v_1,u_2). It is easy to see that $\Psi(H_n') \geq \Psi(H_n)$, and one can therefore assume every vertex in H_n has degree at most three.

If H_n has a vertex of degree one, or a vertex of degree two adjacent to two nonadjacent vertices, then H_n can be reduced as described for Theorem 1, and the value of Ψ would not be changed. We may assume then that H_n has no vertex of degree one and any vertex of degree two is contained in a triangle.

Assuming H_n has a vertex u of degree two, then H_n contains one of the four structures indicated in Figure 2. The following formulas are easily verified.

$$\Psi(G_1') = \Psi(G_1) + \Psi(G_1;a)$$
$$\Psi(G_2') = \Psi(G_2) + 3\cdot\Psi(G_2;a)$$
$$\Psi(G_3') = \Psi(G_3) + \Psi(G_3;a)$$
$$\Psi(G_4') = \Psi(G_4) + k,$$

where k is the number of cycles in G_4 through both a and b. In each case, $\Psi(G_i') \geq \Psi(G_i)$ and p and q are each reduced by one, which

leaves n unchanged. Note that after G_1 is reduced to G_1' that G_1' can be further reduced by a type III or IV reduction, depending upon whether or not w_1 and w_2 are adjacent. Thus all vertices of degree two can be eliminated, H_n can be assumed to only have vertices of degree three, and the proof is complete.

The structures indicated in Figure 1 led us to consider the followi[ng] *pizza* graphs H_m which, as will be discussed later, in general fail to maximize Ψ, but do give us our best general lower bound.

Let H_m be the cubic graph with vertex set $\{v_1,\ldots,v_{2m}\}$, $m \geq 2$, and edge set $E_1 \cup E_2$ where $E_1 = \{v_iv_{i+1}; i=1,\ldots,2m\}$ and $E_2 = \{v_iv_{i+m}; i=1,\ldots,m\}$ (subscripts to be read modulo $2m$). The members of E_2 will be called the *chords* of H_m.

THEOREM 3. *Each member of E_1 is contained in $2^{m-1}+\binom{m}{2}+1$ cycles of H_m, each member of E_2 is contained in $2^{m-1}+2m-2$ cycles of H_m, and $\Psi(H_m) = 2^m+m^2-m+1$.*

Proof. Given a set of k chords, the number of cycles containing this set of chords and no other chords is 2 if $k = 2$ or if k is odd, and 0 otherwise. It follows immediately that H_m has

$$1\binom{m}{0} + 2\binom{m}{1} + 2\binom{m}{2} + \sum_{k=3}^{m}\binom{m}{k}\left(1-(-1)^k\right) = 2^m + m^2 - m + 1$$

cycles.

Furthermore, each chord lies in

$$2\binom{m-1}{0} + 2\binom{m-1}{1} + \sum_{k=2}^{m-1}\binom{m-1}{k}\left(1+(-1)^k\right) = 2^{m-1} + 2m/2$$

cycles.

Now, given a member e of E_1 and any k chords, $1 \leq k \leq m$, there is just one cycle containing e and the chords if k is odd and none if k is even and greater than 2 so that e is contained in

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{1}{2}\sum_{k=3}^{m}\binom{m}{k}\left(1-(-1)^k\right) = 2^{m-1} + \binom{m}{2} + 1$$

cycles.

Since $n = m+1$ in H_m, we obtain the following lower bound for $\Psi(n)$.

COROLLARY 4. $\Psi(n) \geq 2^{n-1} + n^2 - 3n + 3$.

The exact value of $\Psi(n)$ has been determined for $3 \le n \le 8$ by exhaustively counting the total number of cycles in every cubic graph of order $2m$, $2 \le m \le 7$. This counting was facilitated considerably by the valuable tables compiled by Bussemaker et al [1]. The results are presented in Table 1 along with the ratio $\Psi(n)/2^{n-1}$.

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n	$\Psi(n)$	$\Psi(n)/2^{n-1}$	Extremal Graphs
3	7	1.75	$H_2 = K_4$ = 3-cage
4	15	1.88	$H_3 = K_{3,3}$ = 4-cage
5	29	1.81	H_4
6	57	1.78	Petersen = 5-cage
7	109	1.70	#84, #85 of [1]
8	213	1.66	Heawood = 6-cage
13	≥5608	≥1.37	McGee = 7-cage ?
15	≥21200	≥1.30	Coxeter [4] ?
16	≥41400	≥1.26	Tutte - Coxeter = 8-cage ?

Table 1. Some established and conjectured values of $\Psi(n)$.

We believe the lower bound obtained in Corollary 4 to be asymptotically correct in view of the decreasing ratio $\Psi(n)/2^{n-1}$ and other computations to be discussed later.

CONJECTURE 1. $\Psi(n) \sim 2^{n-1}$.

In attempting to reduce the obvious upper bound 2^n-1 we tried to show that $\Psi(n) \le 2\ \Psi(n-1)$ by observing that for any graph G containing an edge e, one has $\Psi(G) = \Psi(G-e) + \Psi(G;e)$. Thus, if one can find an edge e for which $\Psi(G;e) \le \frac{1}{2}\Psi(G)$, then $\Psi(G;e) \le \Psi(G-e) \le \Psi(n-1)$, and consequently for any graph G with $q-p+1 = n$ one has $\Psi(G) = \Psi(G-e) + \Psi(G;e) \le 2\Psi(G-e) \le 2\Psi(n-1)$. Therefore one would have $\Psi(n) \le 2\Psi(n-1)$. If any vertex v has degree at least four, then clearly at least one of the edges incident with v is in at most half of the total number of cycles. This led us to the following conjectures.

CONJECTURE 2. *In any graph* G *other than* K_4 *and* $K_{3,3}$ *with minimum degree* $\delta(G) \ge 3$, *there is an edge* e *with* $\Psi(G;e) \le \Psi(G)/2$.

CONJECTURE 3. *In any graph* G *other than* K_4 *or* $K_{3,3}$ *with minimum degree* $\delta(G) \ge 3$, *for each vertex* v *of* G *there is at least one edge* $e(v)$ *incident with* v *for which* $\Psi(G;e(v)) \le \Psi(G)/2$.

Theorem3 shows that Conjecture 3, if true, is best possible in that two-thirds of the edges of H_m are in (barely) more than half of the cycles so that exactly one of the three edges incident with a vertex is in at most (indeed, less than) half of the cycles.

For n = 3, 4, 5, 6 and 8 there is a unique extremal cubic with $\Psi(n)$ cycles and in each of these cases that unique extremal graph is the (unique) cubic of order $2n-2$ with maximum girth. For $n = 7$ there are two cubics of order 12 that have (maximum) girth 5 and, indeed, both of these graphs and no others have $\Psi(7)$ cycles. Perhaps these circumstances warrant the following conjecture.

CONJECTURE 4. *The cubic graphs of given order with the maximum number of cycles are precisely those with maximum girth.*

If this conjecture is correct, then three computations performed for us by Curtiss A. Barefoot (to whom we express our thanks) are of interest. He determined the number of cycles in the McGee graph, the Coxeter graph, and the Tutte-Coxeter graph; these numbers are included in Table 1 and would lend support to Conjecture 1 given that Conjecture 4 is true.

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