Formulae for some classical constants

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The goal of this paper is to present formulas for Apéry Constant , Archimede's Constant , Logarithm Constant , Catalan's Constant.
Such formulas are useful for high precision calculations frequently appearing in number theory. Also , one motivation for computing digits of some constants is that these are excellent test of the integrity of computer hardware and software.

1. The Apéry's Constant $\zeta(\mathbf{3})$

The Apery's Constant is defined as

$$\zeta(\mathbf{3}) = \sum_{\mathbf{k=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{k^3}} =$$

$$= 1.2020569031595942853997381615114499907649862923404 9... \quad .$$

The designation of $\zeta(3)$ as Apéry's constant is new but well-deserved. In 1979, Roger Apéry stunned the mathematical world with a miraculous proof that $\zeta(3)$ is irrational (see [4], [16]). The above expression is very slow to converge. Another formula is

- **Roger Apéry ([4]-1979)**

$$\zeta(\mathbf{3}) = \frac{\mathbf{5}}{\mathbf{2}} \sum_{\mathbf{k=1}}^{\infty} (-\mathbf{1})^{\mathbf{k-1}} \frac{\mathbf{1}}{\mathbf{k^3}\binom{\mathbf{2k}}{\mathbf{k}}} \tag{1}$$

which was the starting point of Apéry's incredible proof of the irrationality of $\zeta(3)$. Since 1980, many formulas for $\zeta(3)$ were discovered.

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For instance

- T.Amderberhan ([2]-1996) proved that

$$(2)\qquad \zeta(\mathbf{3})=\frac{\mathbf{1}}{\mathbf{4}}\sum_{\mathbf{k=1}}^{\infty}(-\mathbf{1})^{\mathbf{k-1}}\frac{(\mathbf{56k^2-32k+5})\Big((\mathbf{k-1})!\Big)^{\mathbf{3}}}{(\mathbf{2k-1})^{\mathbf{2}}(\mathbf{3k})!}$$

$$(3)\qquad \zeta(\mathbf{3})=\frac{\mathbf{1}}{\mathbf{18}}\sum_{\mathbf{k=1}}^{\infty}(-\mathbf{1})^{\mathbf{k-1}}\frac{(\mathbf{k}!)^{\mathbf{2}}(\mathbf{2k})!\cdot\mathbf{A}(\mathbf{k})}{(\mathbf{4k})!\mathbf{k^3}\cdot\mathbf{B}(\mathbf{k})}$$

with

$$\begin{cases} A(k) & =5265(k-1)^4+13878(k-1)^3+13761(k-1)^2+\\ & +6120(k-1)+1040\\ B(k) & =(3k-1)^2(3k-2)^2\end{cases}$$

- T.Amdeberhan and Doron Zeilberger ([2]-1996), ([3]-1997)

$$(4)\qquad \zeta(3)=\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{Q(k)}{k^5\binom{2k}{k}^5}$$

where

$$Q(k)=205k^2-160k+32\quad .$$

- (A. Lupaş)

$$(5)\qquad \zeta(\mathbf{3})=\frac{\mathbf{3752}}{\mathbf{3125}}+\frac{\mathbf{1}}{\mathbf{6250}}\sum_{\mathbf{k=1}}^{\infty}(-\mathbf{1})^{\mathbf{k-1}}\frac{\mathbf{P}(\mathbf{k})}{\mathbf{k^5}\binom{\mathbf{2k}}{\mathbf{k}}^{\mathbf{5}}(\mathbf{2k+1})^{\mathbf{5}}}$$

with

$$\mathbf{P}(\mathbf{k})=\mathbf{14760k^4+28010k^3+19505k^2+5920k+672}\quad .$$

The proof of (5) was performed by means of WZ-method (see [17] ,[19] also [2] ,[3]) followed by a Kummer transformation.
Let us remind that a discrete function $a(n,k)$ defined for $n,k\in\{0,1,...\}$ is called a *Closed Form =(CF) in two variables* when the ratios $\dfrac{a(n+1,k)}{a(n,k)}$, $\dfrac{a(n,k+1)}{a(n,k)}$ are both rational functions. A pair (F,G) of CF functions which satisfy

$$(6)\qquad F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$$

is a so-called *WZ- form.* There are many investigations connected with such WZ-forms (today known as WZ-theory) : see [2], [3] ,[17] ,[19].
In the 1990's , the WZ theory has been the method of choice in resolving conjectures for hypergeometric identities.The key of the WZ-theory is the following theorem

Theorem 1 (D. Zeilberger-[17], 1993) *For any WZ-form (F,G)*

$$\sum_{k=0}^{\infty} G(k,0) = \sum_{k=0}^{\infty} \Big(\ F(k+1,k) + G(k,k) \ \Big) \ ,$$

whenever either side converges.

Let us remark that all formulas which are listed in (1)-(5)are of the form

(7)
$$\zeta(3) = K_1 + K_2 \sum_{k=1}^{\infty} (-1)^{k-1} a_k \quad , a_k > 0$$

where K_1 , K_2 are some real constants. Supposing that we have also another representation , namely

(8)
$$\zeta(3) = K_1' + K_2' \sum_{k=1}^{\infty} (-1)^{k-1} b_k \quad , b_k > 0$$

we shall say that (7) $\ll$ (8) iff

$$a_n = \mathcal{O}(\zeta_n) \quad , \quad b_n = \mathcal{O}(\eta_n) \quad \text{with} \quad \lim_{n\to\infty} \frac{\zeta_n}{\eta_n} = 0 \ .$$

Formula	$\mathbf{a_n = \mathcal{O}(\zeta_n)}$
(1)	$a_n = \mathcal{O}\Big(\frac{1}{4^n \cdot n^2 \sqrt{n}}\Big)$
(2)	$a_n = \mathcal{O}\Big(\frac{1}{27^n \cdot n^2}\Big)$
(3)	$a_n = \mathcal{O}\Big(\frac{1}{64^n \cdot n^2}\Big)$
(4)	$a_n = \mathcal{O}\Big(\frac{1}{1024^n \cdot \sqrt{n}}\Big)$
(5)	$a_n = \mathcal{O}\Big(\frac{1}{1024^n \cdot n^3 \sqrt{n}}\Big)$

Therefore

$$\boxed{(5) \ll (4) \ll (3) \ll (2) \ll (1) \quad .}$$

G.Fee and Simon Plouffe used (4) in their evaluation of $\zeta(3)$ with 520,000 digits (see [3]). Later, by means of the same formula (4) B.Haible and T.Papanikolau computed $\zeta(3)$ to 1,000,000 digits.

Open problem : If $T_n(x) = \cos(n \cdot \arccos x)$ is the Chebychev's polynomial of the first kind, and (see (5))

$$\begin{cases} P(j) = 14760j^4 + 28010j^3 + 19505j^2 + 5920j + 672 \\ S_k = \dfrac{1}{6250} \sum_{j=1}^{k} (-1)^{j-1} \dfrac{P(j)}{j^5 \binom{2j}{j}^5 (2j+1)^5} \\ \mathcal{A}_n = \dfrac{1}{T_n(3)} \sum_{k=1}^{n} \dfrac{n}{n+k} \binom{n+k}{2k} 4^k S_k \\ \mathcal{B}_{\mathbf{n}} = \mathbf{\frac{3752}{3125}} + \mathcal{A}_{\mathbf{n}} \end{cases} .$$

then prove or disprove that

$$|\zeta(3) - \mathcal{B}_{\mathbf{n}}| = \mathcal{O}\Big(\frac{1}{5939^n \cdot n^3 \sqrt{n}}\Big) \quad , \quad (n \to \infty)$$

The following proposition generalize an identity proved by R.Ap'ery ($x = 0$, [4]) .

Theorem 2 *For* $x \geq 0$

$$\sum_{k=1}^{n} \frac{1}{(k+x)^3} = \frac{5}{2} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^3 \binom{2k+x}{k} \binom{k+x}{k}} \Big(1 - \frac{2x(3k+2x)}{5(k+x)^2}\Big) +$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^k}{k^3 \binom{n+k+x}{k} \binom{n+x}{k}} .$$

A more general function than $\zeta(s)$ is

$$\zeta(s,x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^s} \quad , \mathrm{Re}(s) > 1 \ , \ x > 0 \ .$$

Corollary 1 *If* $x > 0$, *then*

$$\zeta(3, x+1) = \frac{5}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k+x}{k} \binom{k+x}{k}} \Big(1 - \frac{2x(3k+2x)}{5(k+x)^2}\Big) .$$

2. The constants $\ln 2$, $\mathbf{G}$, π

Theorem 3 *Let* $x>0\,,y>0$. *Then*

$$\sum_{k=0}^{\infty}(-1)^k\Big(\ \frac{(x)_k}{(x+2y)_k}\ \Big)^2=$$

(9)

$$=\frac{y}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{16^k}\frac{\Big((2y+1)_{2k}\Big)^3}{\Big((y+1)_k\ (x+2y)_{2k+1})\Big)^2}\frac{10(k+y)^2+(6x+1)(k+y)+x^2}{k+y}\ .$$

Proof. Define

$$A(j)=A(j,x,y)=y\frac{(-1)^j}{16^j}\frac{\Big((2y+1)_{2j}\Big)^3}{\Big((y+1)_j\Big)^2}$$

$$\lambda(n,k)=(-1)^n\Big(\ \frac{(x)_n}{(x+2y)_{n+k}}\ \Big)^2$$

and

$$G(n,k)=\begin{cases}\frac{1}{y+j}\lambda(n,2j)A(j) & ,\quad k=2j\\ \\ 2(y+j+x+n)\lambda(n,2j+1)A(j) & ,\quad k=2j+1\end{cases}\ .$$

Further

$$F(n,k)=\begin{cases}\frac{\lambda(n,2j)}{2(y+j)}A(j) & ,\quad k=2j\\ \\ (n+x+3y+3j+\frac{1}{2})\lambda(n,2j+1)A(j) & ,\quad k=2j+1\end{cases}\ .$$

If $G(n,k)$ and $F(n,k)$ are defined as above , then

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)\quad ,$$

that is (F,G) is a WZ-form. By applying WZ-method we find (9).
Further, by G is denoted the Catalan constant G which is defined as

$$G=\beta(2)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}=0.91596554...\ ,$$

$\beta(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^x}$ being the Dirichlet's beta function .
Some particular cases of (9) are

x	y	**Identity**	$\mathbf{a_k}$	The order of a_n $(n \to \infty)$
$\frac{1}{2}$	1	$\pi = 4 - \sum_{k=1}^{\infty} (-1)^{k-1} a_k$	$\frac{\binom{2k}{k}}{\binom{4k}{2k}^2} \frac{40k^2+16k+1}{2k(4k+1)^2}$	$\mathcal{O}\left(\frac{1}{64^n \cdot \sqrt{n}}\right)$
1	1	$\ln 2 = \frac{3}{2} - \sum_{k=1}^{\infty} (-1)^{k-1} a_k$	$\frac{1}{16^k} \binom{2k}{k} \frac{5k+1}{k\left(k+\frac{1}{2}\right)}$	$\mathcal{O}\left(\frac{1}{4^n \cdot n^{\frac{3}{2}}}\right)$
$\frac{1}{2}$	$\frac{1}{2}$	$G = \sum_{k=1}^{\infty} (-1)^{k-1} a_k$ (Catalan' s Constant)	$\frac{2^{8k} \cdot (40k^2-24k+3)}{k^3 \cdot \binom{4k}{2k}^2 \binom{2k}{k} 64(2k-1)}$	$\mathcal{O}\left(\frac{1}{14^n \cdot \sqrt{n}}\right)$
1	$\frac{1}{2}$	$\pi^2 = \sum_{k=1}^{\infty} (-1)^{k-1} a_k$	$\frac{1}{k^3 \binom{2k}{k}} \cdot \frac{14k^2-9k+2}{3(2k-1)}$	$\mathcal{O}\left(\frac{1}{4^n \cdot n^{\frac{3}{2}}}\right)$

The mathematicians have assumed that there is no shortcut to determining just the $n-$th digit of π . Thus it came as a great surprise when such a scheme was in 1997 discovered [6] (see (11) from below). This formula was discovered empirically, using months of PSLQ computations , see [8]. This is likely the first instance in history that a signifiant new formula for π was discovered by a computer. In the following, we present an identity which gives a proof of (11).

Proposition 1 *If* $|z+1| < \sqrt{2}$, $r \in \mathbf{C}$, *then*

$$\pi + 4 \arctan z + (2+8r) \ln \frac{1-2z-z^2}{1+z^2} = (4+8r) \sum_{k=0}^{\infty} \frac{(1+z)^{8k+1}}{8k+1} \frac{1}{16^k} -$$

$$(10) \qquad - 8r \sum_{k=0}^{\infty} \frac{(1+z)^{8k+2}}{8k+2} \frac{1}{16^k} - 4r \sum_{k=0}^{\infty} \frac{(1+z)^{8k+3}}{8k+3} \frac{1}{16^k} -$$

$$- (2+8r) \sum_{k=0}^{\infty} \frac{(1+z)^{8k+4}}{8k+4} \frac{1}{16^k} - (1+2r) \sum_{k=0}^{\infty} \frac{(1+z)^{8k+5}}{8k+5} \frac{1}{16^k} -$$

$$- (1+2r) \sum_{k=0}^{\infty} \frac{(1+z)^{8k+6}}{8k+6} \frac{1}{16^k} + r \sum_{k=0}^{\infty} \frac{(1+z)^{8k+7}}{8k+7} \frac{1}{16^k} .$$

A proof of the above asertion is given in [15].

If in (10) we select $(z,r)=(0,-\frac{1}{2})$ then

$$\pi=\sum_{k=0}^{\infty}\Big(\frac{4}{8k+2}+\frac{2}{8k+3}+\frac{2}{8k+4}-\frac{1}{2(8k+7)}\Big)\frac{1}{16^k}\ .$$

Likewise, using the identity

$$\arctan 1+\arctan z=\arctan t\ ,\ t\ :=\frac{1+z}{1-z}\ ,(z<1)\ .$$

one finds

$$\pi=\sum_{k=0}^{\infty}\Big(\frac{2}{8k+1}+\frac{2}{8k+2}+\frac{1}{8k+3}-\frac{1}{2(8k+5)}-\frac{1}{2(8k+6)}-\frac{1}{4(8k+7)}\Big)\frac{1}{16^k}\ .$$

Let us remark that for $(z,r)=(0,0)$ we find from (10) the remarkable **BBP formula** (attributed to David **B**ailey , Peter **B**orwein and Simon **P**louffe) for π , that is

$$\pi=\sum_{k=0}^{\infty}\Big(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\Big)\frac{1}{16^k}\quad . \tag{11}$$

Also when $(z,r)=(0,r), r\in \mathbf{C}$ one finds another formula for π , namely

$$\pi=$$

$$=\sum_{k=0}^{\infty}\Big(\frac{4+8r}{8k+1}-\frac{8r}{8k+2}-\frac{4r}{8k+3}-\frac{2+8r}{8k+4}-\frac{1+2r}{8k+5}-\frac{1+2r}{8k+6}+\frac{r}{8k+7}\Big)\frac{1}{16^k}$$

which was discovered by Victor Adamchik and Stan Wagon in their interesting HTML paper [1] .

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