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Comb. Maths IX (Brisbane, 1981) ed EJB, SO-W, APS.
Springer-Verlag LNiM 952 (1982)
E.J Billington
Sheila Oates-Williams
& APS

SEQUENTIAL BINARY ARRAYS II: FURTHER RESULTS ON THE SQUARE GRID

ANNE PENFOLD STREET AND ROBERT DAY

A periodic binary array is said to be *sequential* if and only if every line of the array is occupied by a given periodic binary sequence, or by some cyclic shift or reversal of this sequence. Such arrays are of interest in connection with experimental layouts. In this paper, we extend in two ways our earlier work on arrays built on the square grid: first, by enumerating the equivalence classes of sequential arrays with sequence 0^n1; secondly, by studying further properties of sequential arrays with sequences 10111000100 and 111011001010000, that is, the incidence sequences of the difference sets of integers modulo 11 and 15 respectively.

1. INTRODUCTION

A periodic binary sequence $\alpha_n = \{a_i\}$ of period n is a sequence of zeros and ones such that n is the smallest positive integer for which

$$a_i = a_{i+n} \quad \text{for all } i.$$

A periodic binary array $A_n = \{a_{ij}\}$ of period n on the square grid is an array each of whose rows and columns is a periodic binary sequence of period n. Such an array is said to be *sequential* if the same sequence (or its shifts or reversals) occurs in every row and column. Sequential arrays on the square grid (and also on triangular and hexagonal grids) are of interest in connection with some problems in agricultural statistics [3,5,6,8,9]. Examples of sequential arrays are shown in Figure 1. Any array of period n on the square grid may be regarded as consisting of repetitions of an $n \times n$ matrix: in particular, if the array is sequential, then its corresponding matrix has the same sequence (or its cyclic shifts or reversals) in every row and column and will also be called sequential.

We shall consider two binary arrays to be equivalent if one can be obtained from the other by interchanging zeros with ones, or by rotation or reflection, or by some finite sequence of these operations.

We shall also consider two $n \times n$ binary matrices to be equivalent if they generate equivalent binary arrays; see Figure 2.

0	0	1	1	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	1	1	0	0

0	0	1	1	0	0	1	1	0	0	1	1
1	0	0	1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1	1	0	0
0	1	1	0	0	1	1	0	0	1	1	0
0	0	1	1	0	0	1	1	0	0	1	1
1	0	0	1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1	1	0	0
0	1	1	0	0	1	1	0	0	1	1	0

0	0	1	1
0	0	1	1
1	1	0	0
1	1	0	0

0	0	1	1
1	0	0	1
1	1	0	0
0	1	1	0

Figure 1: Sections of two sequential arrays with sequence 0 0 1 1, of period 4, and their corresponding matrices.

0	0	1	1		0	0	1	1		0	1	1	0		0	1	1	0		1	1	0	0
0	0	1	1		1	1	0	0		1	0	0	1		0	1	1	0		1	1	0	0
1	1	0	0		1	1	0	0		1	0	0	1		1	0	0	1		0	0	1	1
1	1	0	0		0	0	1	1		0	1	1	0		1	0	0	1		0	0	1	1

Figure 2: Some equivalent binary matrices, sequential with sequence 0 0 1 1.

Two binary sequences of length n are said to be *necklace equivalent* if and only if one can be obtained from the other by interchanging zeros and ones, or by a cyclic shift or by reversal, or by some finite sequence of these operations. Thus, the equivalence classes are determined by the action of the group $D_{2n} \times S_2$, the direct product of the dihedral group of order $2n$ with the symmetric group of degree two [4]. A binary sequence is said to be *self-complementary* if and only if it can be obtained from its complement by cyclic shift, or by reversal, or by some finite sequence of these operations, that is, under the action of D_{2n} alone.

Hence in order to generate all inequivalent binary sequential arrays of period n, we start from a complete set of representatives of necklace equivalence classes of binary sequences of length n; from each binary sequence in this set we generate the inequivalent sequential $n \times n$ matrices. Two matrices are regarded as equivalent if one can be obtained from the other by a cyclic shift with respect to rows or columns, by rotation, by transposition, by complementation, or by any finite sequence of these operations. Thus, if we let u denote the cyclic rotation of rows that takes row i to row $i-1$ (modulo n), v the corresponding rotation of columns, w the rotation of the matrix clockwise through a right-angle, and x transposition, then the equivalence classes are determined by the action of the group $H = G \times S_2$, where for $n \geq 3$,

$$G = \langle u,v,w,x \mid u^n = v^n = w^4 = x^2 = 1,\ uv = vu,\ uw = wv,\ ux = xv,$$
$$xw = w^3x,\ vw = wu^{-1},\ vx = xu\rangle.$$

None of the sequences considered in this paper is self-complementary, so we are concerned only with the group G, rather than the whole of H.

We use the notation a^k to denote a string of k copies of the symbol a; thus $0^3 1^2$ denotes 00011. For the sequences to which we refer most frequently, we use the following notation:

$$\pi_n = 0^{n-1}1; \quad \delta_{11} = 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0;$$

$$\delta_{15} = 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0.$$

Note that δ_{11} and δ_{15} are the incidence sequences of the difference sets of integers modulo 11 and 15 respectively.

Any sequential matrix having π_n as its sequence is a permutation matrix, and is denoted by P_n. As usual, 0_n denotes the $n \times n$ matrix with all entries zero, and J_n the $n \times n$ matrix with all entries one.

We use the term *m-step circulant* to mean an $n \times n$ matrix such that

$$a_{i,j+m} = a_{i-1,j}$$

for all $i,j = 1,2,\ldots,n$, for some fixed m, where subscripts are added modulo n. Thus a 1-step circulant is just the usual circulant. Further, we use the term "circulant" to include back-circulant matrices, since the two are equivalent under the action of the group G.

In Section 2, we enumerate the equivalence classes of permutation arrays of period n, using Burnside's Lemma. In Section 3, we discuss proper[illegible] of sequential arrays with sequences δ_{11}, and δ_{15}, thu[illegible] extend[illegible] results in [2].

2. COUNTING INEQUIVALENT PERMUTATION ARRAYS

The number of equivalence classes is most easily calculated from Burnside's Lemma.

Lemma (Burnside [1, p.191]). *Let Γ be a finite group, of order g, of permutations acting on a finite set S, and let two elements of S be equivalent if and only if one can be transformed into the other by a permutation in Γ. Then the number, T, of inequivalent elements is*

$$T = \frac{1}{g} \sum_{t \in \Gamma} I(t),$$

where $I(t)$ is the number of elements of S left invariant by the permutation $t \in \Gamma$, and the sum is over all g permutations in Γ.

Now let N be the normal subgroup of the group G generated by cyclic shifts, that is, $N = \langle u,v\rangle \cong Z_n \times Z_n$. We may conveniently regard G as the union of the eight cosets N, Nw, Nw^2, Nw^3, Nx, Nwx, Nw^2x, Nw^3x. Table 1 gives the contribution of each coset to the value of $8n^2T(n)$, where $T(n)$ is the number of equivalence classes of permutation arrays of period n. Thus we have

Theorem 1. *The value of $T(n)$ can be calculated by summing the appropriate values in Table 1, and dividing by $8n^2$.*

Proof. The proof is an application of Burnside's Lemma to the set S of permutation arrays of period n, where the group Γ is G, defined in Section 1, of order $g = 8n^2$.

The rows and columns of the permutation matrix P_n are labelled from 0 to $n-1$, and the generating transformations u, v, w, x, of G act on the elements of P_n as follows:

$$(i,j)u = (i-1,j);$$
$$(i,j)v = (i,j-1);$$
$$(i,j)w = (j,n-1-i);$$
$$(i,j)x = (j,i),$$

where all operations are carried out modulo n.

A set of n elements of P_n, one from each row and one from each column, will be called a *transversal* of P_n, and a set of $k < n$ elements of P_n, with at most one from each row and at most one from each column, will be called *extendable*. A transversal which consists of elements $\{(i,j)\}$, such that either $i+j$ is [illegible]tant or $i-j$ is constant will be called a *diagonal* of P_n.

Coset	Contribution to $8n^2.T(n)$
N	$\sum_{d\mid n} \{\phi(n/d)\}^2.(n/d)^d\ d!$
Nw^2	$\begin{cases} 2^h.(h+1)!h & n = 2h, \\ 2^h.h!(2h+1)^2 & n = 2h+1. \end{cases}$
Nw	$\begin{cases} 2^{k+3}.k^2.\prod_{\ell=1}^{k}(2\ell-1), & n = 4k \\ 2^k.(4k+1)^2.\prod_{\ell=1}^{k}(2\ell-1), & n = 4k+1 \\ 2^{k+2}.(2k+1)^2.\prod_{\ell=1}^{k}(2\ell-1), & n = 4k+2 \\ 0, & n = 4k+3 \end{cases}$
Nw^3	As for Nw.
Nx	$\begin{cases} \sum_{d\mid n} d!n\phi(n/d).\sum_{\ell=0}^{\lfloor d/2\rfloor}\frac{n^\ell}{(2d)^\ell.\ell!(d-2\ell)!}, & n \text{ odd}, \\ \sum_{d\mid n,\ n/d \text{ odd}} d!n\phi(n/d).\sum_{\ell=0}^{\lfloor d/2\rfloor}\frac{n^\ell}{(2d)^\ell.\ell!(d-2\ell)!}, & n = 4m+2, \\ \sum_{d\mid n,\ n/d \text{ odd}} d!n\phi(n/d).\sum_{\ell=0}^{\lfloor d/2\rfloor}\frac{n^\ell}{(2d)^\ell.\ell!(d-2\ell)!} + \sum_{d\mid n,\ n/d \text{ even},\ d=2h} d!n\phi(n/d)\frac{n^h}{(2d)^h h!}, & n = 4m. \end{cases}$
Nw^2x	As for Nx.
Nwx	$\begin{cases} 2^h.h!h, & n = 2h \\ 0, & n = 2h+1 \end{cases}$
Nw^3x	As for Nwx.

Table 1: see Theorem 1.

We consider the elements of G, coset by coset. Note that for $g \in G$, $|I(g)|$ is the number of transversals of an $n\times n$ array fixed setwise by g. We let α and β denote integers modulo n.

Coset N has typical element $u^\alpha v^\beta$, with action

$$(i,j)u^\alpha v^\beta = (i-\alpha, j-\beta).$$

Let $d_1 = \gcd(\alpha,n)$ and $d_2 = \gcd(\beta,n)$. Then the order of $u^\alpha v^\beta$ is k, where

$$k = \operatorname{lcm}[n/d_1,\ n/d_2].$$

The typical orbit is then

$$(i,j),(i-\alpha,j-\beta),(i-2\alpha,j-2\beta),\ldots,(i-(k-1)\alpha,j-(k-1)\beta).$$

If $d_1 \neq d_2$, then

$$(i,j)(u^\alpha v^\beta)^{n/d_1} = (i,j-(n/d_1)\beta)$$

and $(n/d_1)\beta \not\equiv 0 \pmod n$.

Hence the orbit contains at least two distinct elements from the same row, and cannot be extendable.

If $d_1 = d_2 = d$, then the square is partitioned into nd orbits, each containing n/d elements, and any of these orbits is extendable. A transversal fixed by $u^\alpha v^\beta$ must be a union of d orbits.

In order to count the number of ways to choose such a transversal, we observe that the row indices of all the elements of one particular orbit form the entire congruence class, modulo d, of the integers modulo n. A similar statement holds true for the column indices. Hence the $n\times n$ square is partitioned into $n/d \times n/d$ subsquares, each containing n/d orbits, and each consisting of elements (i,j), where $i \equiv e$, $j \equiv f \pmod d$. To choose a transversal we first choose d subsquares with row indices congruent to $0,1,2,\ldots,d-1 \pmod d$; this choice may be made in $d!$ ways, since any congruence class of column indices may be associated with each class of row indices. Within each subsquare, we choose one of its n/d orbits, and we have d subsquares. Hence the total number of transversals possible, for given α and β, is

$$(n/d)^d.d!.$$

Since α and β may each be chosen independently in $\phi(n/d)$ ways, the total contribution to $8n^2.T(n)$ is

$$\sum_{d\mid n}\{\phi(n/d)\}^2.(n/d)^d.d!.$$

An example may help to make this clearer. If $n = 6$, $\alpha = 4$, $\beta = 3$, then $d_1 = 2$, $d_2 = 3$, $k = 6$. A typical orbit is (0,0), (2,3), (4,0), (0,3), (2,0), (4,8); other orbits are translates of this one. Since each orbit contains more than one element in some rows and some columns, it cannot be extended to a transversal. But if $n = 6$, $\alpha = 4$, $\beta = 2$, then $d_1 = d_2 = d = 2$, $k = 3$. A typical orbit is (0,0), (2,4), (4,2); the square is partitioned into 12 orbits, each a translate of this one. We may also regard the square as being partitioned into four subsquares, where the row and column indices are respectively even-even, even-odd, odd-even, odd-odd, and each of these subsquares is partitioned into three orbits. To obtain a transversal, we choose either (i) the even-even and odd-odd subsquares,

or (ii) the even-odd and odd-even subsquares.

From each of the two chosen subsquares we choose an orbit in three ways. Hence the total number of transversals fixed under u^4v^2 is $(6/2)^2.2!$.

Counting arguments for the other cases are similar, and we deal with them more briefly.

<u>Coset Nw^2</u> has typical element $u^\alpha v^\beta w^2$, with action

$$(i,j)u^\alpha v^\beta w^2 = (\alpha-1-i, \beta-1-j).$$

The typical orbit is

$$(i,j),\ (\alpha-1-i, \beta-1-j),$$

which reduces to a fixed point when

$$2i = \alpha-1 \quad \text{and} \quad 2j = \beta-1, \text{ modulo } n.$$

(i) Let $n = 2h$.

If α and β are both even, any 2-cycle is extendable and no fixed point occurs. A fixed transversal must consist of h 2-cycles, and may be chosen in $2^h.h!$ ways. Since α and β may each be chosen independently in h ways, we have a contribution to $8n^2.T(n)$ of

$$h^2.2^h.h! \qquad (1)$$

If α is odd and β is even, no fixed point occurs, and any 2-cycle is extendable except for those of the form (i_1,j), $(i_1,\beta-1-j)$ and (i_2,j), $(i_2,\beta-1-j)$ where $2i_1 = 2i_2 = \alpha-1$. Hence no fixed transversal is possible.

If α is even and β is odd, the non-extendable 2-cycles are those in columns j_1 and j_2, where $2j_1 = 2j_2 = \beta-1$. A similar argument shows that no fixed transversal is possible.

If α and β are both odd, any of the fixed points (i_1,j_1),

(i_1,j_2), (i_2,j_1), and (i_2,j_2) is extendable, and so is any of the 2-cycles disjoint from rows i_1, i_2 and columns j_1, j_2. Hence a fixed transversal must consist of two fixed points and $(h-1)$ 2-cycles, and may be chosen in $2^h.(h-1)!$ ways. As before, we have a contribution to $8n^2.T(n)$ of

$$h^2.2^h.(h-1)!. \qquad (2)$$

Adding (1) and (2) gives a total contribution to $8n^2.T(n)$ of

$$2^h.(h+1)!h, \quad \text{for } n = 2h.$$

(ii) Let $n = 2h+1$.

For any α and β, let i' and j' be the unique solutions to $2i = \alpha-1$ and $2j = \beta-1$. The non-extendable 2-cycles are those in row i' and column j'. The unique fixed point (i',j'), and any other 2-cycle are extendable. A fixed transversal consists of (i',j') and h 2-cycles, and may be chosen in $2^h.h!$ ways. Since α and β may each be chosen independently in $2h+1$ ways, the total contribution to $8n^2.T(n)$ is

$$(2h+1)^2.2^h.h!, \quad \text{for } n = 2h+1.$$

<u>Coset Nw</u> has typical element $u^\alpha v^\beta w$, with action

$$(i,j)u^\alpha v^\beta w = (j-\beta, \alpha-1-i).$$

The typical orbit is the 4-cycle (i,j), $(j-\beta,\alpha-1-i)$, $(\alpha-\beta-1-i, \alpha+\beta-1-j)$, $(\alpha-1-j, i+\beta)$, which in some cases reduces to a 2-cycle or to a fixed point.

If n is even, and $\alpha \pm \beta$ is even, no 2-cycle nor fixed point can occur. If n is even and $\alpha \pm \beta$ is odd, there are two fixed points and one 2-cycle. These are found from the solutions i_1 and i_2 of $2i = \alpha-\beta-1$. Since $i_2 = i_1 + n/2$, the fixed points are $(i_1, i_1+\beta)$ and $(i_1 + n/2,\ i_1 + n/2 + \beta)$ and the 2-cycle is $(i_1,\ i_1 + n/2 + \beta)$, $(i_1 + n/2,\ i_1 + \beta)$.

If n is odd, no 2-cycle can occur, but for each choice of α and β, we have the unique fixed point $(i', i'+\beta)$ where $2i' = \alpha-\beta-1$.

(i) Let $n = 4k$.

If $\alpha \pm \beta$ is odd, no 4-cycle intersecting row i_1 is extendable, so any fixed transversal would have to include the two fixed points and the 2-cycle. But this again includes two points in row i_1, so no fixed transversal is possible.

If $\alpha \pm \beta$ is even, any fixed transversal consists of k 4-cycles, all disjoint from the diagonals $(j-\beta, j)$ and $(\alpha-1-j, j)$. Hence they can be chosen in $2^k.\prod_{\ell=1}^{k}(2\ell-1)$ ways. Since α and β can be chosen in $\frac{1}{2}n^2 = 8k^2$ ways, the total contribution to $8n^2(T(n)$ is

$$2^{k+3}.k^2.\prod_{\ell=1}^{k}(2\ell-1), \quad \text{for } n = 4k.$$

(ii) Let $n = 4k+2$.

If $\alpha \pm \beta$ is even, no fixed transversal is possible, since it would have to consist of 4-cycles, and $4 \nmid n$.

If $\alpha \pm \beta$ is odd, a fixed transversal must consist of k 4-cycles, together with either two fixed points or one 2-cycle. The two special points are in rows i_1, $i_1 + n/2$ and columns $i_1+\beta$, $i_1 + n/2 + \beta$. The 4-cycles are chosen to avoid these rows and columns, and to avoid the diagonals through the two fixed points. Hence there are $2^{k+1}.\prod_{\ell=1}^{k}(2\ell-1)$ possible fixed transversals. Since α and β can be chosen in $\frac{1}{2}n^2 = 2(2k+1)^2$ ways, the total contribution to $8n^2.T(n)$ is

$$2^{k+2}.(2k+1)^2.\prod_{\ell=1}^{k}(2\ell-1), \quad \text{for } n = 4k+2.$$

(iii) Let $n = 4k+1$.

A fixed transversal must consist of the unique fixed point $(i',i'+\beta)$, together with k 4-cycles, chosen to avoid row i' and column $i'+\beta$, and to avoid the diagonals $(j-\beta,j)$ and $(\alpha-1-j,j)$. Hence the 4-cycles can be chosen in $2^k.\prod_{\ell=1}^{k}(2\ell-1)$ ways, and since any choices of α and β are permissible, the total contribution to $8n^2.T(n)$ is

$$(4k+1)^2.2^k.\prod_{\ell=1}^{k}(2\ell-1), \quad \text{for } n = 4k+1.$$

(iv) Let $n = 4k+3$.

No transversal can be made up of one fixed point and a set of 4-cycles.

<u>Coset Nx</u> has typical element $u^\alpha v^\beta x$, with action

$$(i,j)u^\alpha v^\beta x = (j-\beta, i-\alpha).$$

This leads to a cycle of length dividing $2n$, since the orbit has

(i,j) at step 0,

$(j-\beta, i-\alpha)$ at step 1,

$(i-(\alpha+\beta), j-(\alpha+\beta))$ at step 2,

and in general

$(j-(h-1)(\alpha+\beta)-\beta, i-(h-1)(\alpha+\beta)-\alpha)$ at step $2h-1$,

$(i-h(\alpha+\beta), j-h(\alpha+\beta))$ at step $2h$.

Let $\gcd(\alpha+\beta,n) = d$. Two cases arise.

(i) If $n/d = 2k-1$, then there are essentially three kinds of cycles, as follows:

the diagonal $i = j-(k-1)(\alpha+\beta)-\beta$ is partitioned into d cycles, each of length n/d, each extendable since it is contained in a diagonal;

the $2k-2$ diagonals

$$i = j-h(\alpha+\beta)-\beta, \quad h = 0,1,\ldots,2k-2, \quad h \neq k-1,$$

are partitioned into $(n-d)/2$ cycles, each of length $2n/d$, all of which contain repeated elements from some rows or columns and are hence not extendable;

the remaining $n-(n/d)$ diagonals are partitioned into $n(d-1)/2$ cycles, each of length $2n/d$, all of which are extendable.

A transversal fixed by $u^\alpha v^\beta x$ must consist of a union of ℓ of the extendable $(2n/d)$-cycles, together with $d-2\ell$ of the (n/d)-cycles, where $0 \le 2\ell \le d$. For given ℓ, we may choose these (n/d)-cycles in $\binom{d}{2\ell}$ ways. The rows and columns that intersect the 2ℓ rejected (n/d)-cycles must now be covered by the ℓ $(2n/d)$-cycles. To complete the counting argument, we count the number of ways that $(2n/d)$-cycles could be constructed from the elements of these 2ℓ (n/d)-cycles.

First, we group the 2ℓ (n/d)-cycles into ℓ pairs; since the ordering of the pairs is unimportant, this can be done in $\frac{(2\ell)!}{(2!)^\ell.\ell!}$ ways. Secondly, if the two cycles in a pair are given by

$$(i_1,j_1),(i_2,j_2),\ldots,(i_{n/d},j_{n/d})$$

and

$$(i_1',j_1'),(i_2',j_2'),\ldots,(i_{n/d}',j_{n/d}'),$$

then they may be made to correspond in n/d ways, so that the starting point of the $(2n/d)$-cycle can be taken as

$$(i_1,j_1') \text{ or } (i_1,j_2') \text{ or } \ldots \text{ or } (i_1,j_{n/d}').$$

Hence the ℓ pairs of (n/d)-cycles lead to a choice of $(n/d)^\ell$ $(2n/d)$-cycles.

Thus for each divisor d of n, such that n/d is odd, the number of possible choices is $\frac{d}{2^\ell.\ell!(d-2\ell)!}\left(\frac{n}{d}\right)^\ell$, for each ℓ with $0 \le 2\ell \le d$. Since for given d, we have $n.\phi(n/d)$ choices of α and β, we have a contribution to $8n^2.T(n)$ of

$$\sum_{d|n} d!n\phi(n/d) \sum_{\ell=0}^{\lfloor d/2 \rfloor} \frac{n^\ell}{(2d)^\ell.\ell!(d-2\ell)!} \qquad (3)$$

for n/d odd. Hence for n odd, this is the total contribution.

(ii) If $n/d = 2k$, then there are $nd/2$ $(2n/d)$-cycles. For a union of their orbits to form a transversal, we must have $(2n/d)|n$, that is, $2|d$. Hence for $n \equiv 2 \pmod 4$, no such divisor d exists, and the value given in (3) above is again the total contribution to $8n^2.T(n)$.

If $n \equiv 0 \pmod 4$, let $d = 2h$. This time there are two kinds of cycles, as follows:

the $2k$ $(= n/d)$ diagonals

$$i = j-\ell(\alpha+\beta)-\beta, \quad \ell = 0,1,\ldots,2k-1,$$

are partitioned into $n/2$ cycles, each of length $2n/d$, all of which contain repeated elements from some rows or columns and are hence not extendable;

the remaining $n-(n/d)$ diagonals are partitioned into $n(d-1)/2$ cycles, each again of length $2n/d$, all of which are extendable.

A transversal fixed by $u^\alpha v^\beta x$ must consist of a union of $d/2$ of the extendable $(2n/d)$-cycles. We count the number of such transversals in the following way. The permutation partitions the set of all rows into d $(= 2h)$ cyclically-ordered sets, each of n/d elements, namely

$$i,\ i-(\alpha+\beta),\ i-2(\alpha+\beta),\ \ldots,\ i-(2k-1)(\alpha+\beta), \quad \text{for } i = 0,1,\ldots,d-1.$$

(It also acts in the same way on the columns.) We pair these sets of rows into h sets of pairs; since the ordering of the pairs is unimportant, this may be done in $\frac{(2h)!}{(2!)^h h!}$ ways. Each pair of these cyclic sets of rows may be made to correspond in n/d ways, thus the sets

$$i_1,\ i_1-(\alpha+\beta),\ i_1-2(\alpha+\beta),\ \ldots,\ i_1-(2k-1)(\alpha+\beta)$$

and

$$i_2,\ i_2-(\alpha+\beta),\ i_2-2(\alpha+\beta),\ \ldots,\ i_2-(2k-1)(\alpha+\beta)$$

may be interleaved beginning $i_1,\ i_2,\ i_1-(\alpha+\beta),\ i_2-(\alpha+\beta),\ \ldots$ or $i_1,\ i_2-(\alpha+\beta),\ i_1-(\alpha+\beta),\ i_2-2(\alpha+\beta),\ \ldots$ and so on. The relative ordering of the rows then determines the columns, for if we have the sequence $i_1,i_2,\ldots$ of rows, we must start from the point $(i_1,i_2+\beta)$. Since we have h pairs of cycles to interleave, we have $(n/d)^h$ choices here.

Hence we have

$$\frac{(2h)!}{(2!)^h h!}(n/d)^h = \frac{d!n^h}{(2d)^h.h!}$$

different fixed transversals. (Notice here that h is fixed for given d, whereas in the previous case with n/d odd, ℓ could vary.)

Again α and β can be chosen in $n.\phi(n/d)$ ways, giving an additional contribution to $8n^2.T(n)$, for $n \equiv 0 \pmod 4$, of

$$\sum_{\substack{d|n \\ n/d \text{ even},\ d=2h}} d!n.\phi(n/d)\frac{n^h}{(2d)^h.h!}. \qquad (4)$$

<u>Coset Nwx</u> has typical element $u^\alpha v^\beta wx$, with action

$$(i,j)u^\alpha v^\beta wx = (\alpha-1-i, j-\beta).$$

This leads to a cycle of the form

$$(i,j),\ (\alpha-1-i,j-\beta),\ (i,j-2\beta),\ (\alpha-1-i,j-3\beta)\ \ldots,$$

which is extendable only if it is in fact a fixed point or a 2-cycle.

For a fixed point, we must have $2i = \alpha-1$, $\beta = 0$; for a 2-cycle, we must have $\beta \ne 0$, $2\beta = 0$. If $\beta = 0$, we can have no fixed transversal. Otherwise we must have $n = 2k$, $\beta = k$, and α even, so that no row is fixed. For given α, we can find $k!2^k$ fixed transversals; since for $n = 2k$, α can be chosen in k ways, we have a contribution to $8n^2.T(n)$, of $k.2^k.k!$, for $n = 2k$.

But for n odd, no fixed transversal is possible.

<u>Coset Nw^3</u> gives the same contribution as Nw; similarly Nw^2x gives the same as Nx, and Nwx the same as Nw^3x.

This completes the proof. □

Values for $T(n)$ are given in Table 2. They were calculated in two ways: from Theorem 1, and by generating all inequivalent arrays as described in [2].

This sequence appears to be new, in the sen f being not listed in Sloane [7].

A684-1

n	T(n)
1	1
2	1
3	1
4	2
5	4
6	10
7	28
8	127

Table 2.

3. SEQUENTIAL MATRICES BASED ON DIFFERENCE SETS

In [2], we characterised certain sequential matrices in terms of the properties of their sequences. If $D \subseteq Z_n$, the integers modulo n, then the incidence sequence δ of D is defined by

$$\delta = d_0, d_1, \ldots, d_{n-1}$$

where

$$d_i = \begin{cases} 1, & i \in D, \\ 0, & \text{otherwise}. \end{cases}$$

From [2, Theorem 2] and its corollaries, we can find the values of m such that m-step circulant sequential arrays exist, when D is a difference set in Z_n. However, many non-circulant sequential arrays exist for δ_n, the incidence sequence of such a difference set.

Some sets of these arrays are related to each other in the following way: if A is an $n \times n$ sequential matrix with sequence δ_n, the incidence sequence of a difference set, then consider the matrix B, defined by

$$b_{ij} = a_{1+(i-1)m,j}, \quad \text{for } i,j = 1,2,\ldots,n.$$

If m is an element of the multiplier group of D, then B is also sequential, with sequence δ_n. For n = 7 or 11 and D the quadratic residues, the symmetry group of B is the same as that of A, but for larger n, this property no longer holds.

Arrays for δ_7 were listed in [2, Section 5]. Those for δ_{11} are given in Table 3, and those for δ_{15} in Figure 3 and Table 4. In Table 3, the notation "1,7,5,3,..." for the class 1 representative denotes an array with the sequence starting in position 1 of row 1, position 7 of row 2, position 5 of row 3, and so on. We note also that the only sequential arrays for the quadratic residues modulo 13 and 17 are the m-step circulants.

Class Number	Class representative	Number per class	Symmetries of array
1	1,7,5,3,8,11,6,10,2,9,4	22	x
2	1,6,7,2,4,3,5,8,11,9,10	22	w^2x
3	1,6,3,2,8,7,11,4,5,10,9	22	x
4	1,3,11,7,4,5,6,9,2,10,8	22	x
5	1,3,5,7,9,11,2,4,6,8,10	4	uv^2
6	1,7,2,6,4,8,9,11,10,3,5	22	w^2x
7	1,2,3,4,5,6,7,8,9,10,11	2	uv, w^2x
8	1,5,9,2,6,10,3,7,11,4,8	4	u^2v^3

Table 3. Sequential arrays with sequence δ_{11} = 10111000100.

The arrays for δ_{11} fall into eight equivalence classes; the class representatives given in Table 3 were the earliest representatives for each of the classes found by the algorithm given in [2]. Each of the arrays has at least one non-trivial symmetry, and symmetries which generate its group are listed in the last column.

The arrays for δ_{15} fall into 64 equivalence classes, which are shown in Figure 3. Each array is numbered m-n, where m denotes the number of the equivalence class, and n the number of the first matrix in that class, as generated by the algorithm of [2]. The symmetry groups of these arrays are listed in Table 4.

Array	Symmetries	Group order
2-2	uv^{-1}, x	30
61-323	uv^4, w^2x	30
63-464	u^2v^{-1}	15
31-68	u^5v^{-5}, x	6
53-214	u^5v^5, w^2x	6
62-343	u^5v^5, x	6
64-707	u^5v^{-5}, w^2x	6
1-1, 3-3, 4-4, 5-7, 6-8, 8-10, 9-11, 12-14, 15-18, 19-24, 20-33, 22-37, 24-40, 28-45, 33-72, 34-74, 35-75, 36-78, 37-81, 44-103, 45-108, 47-166	x	2
49-170, 52-212, 55-233, 60-256	w^2x	2

Table 4. Symmetries of the arrays in Figure 3. All arrays not listed have trivial group.

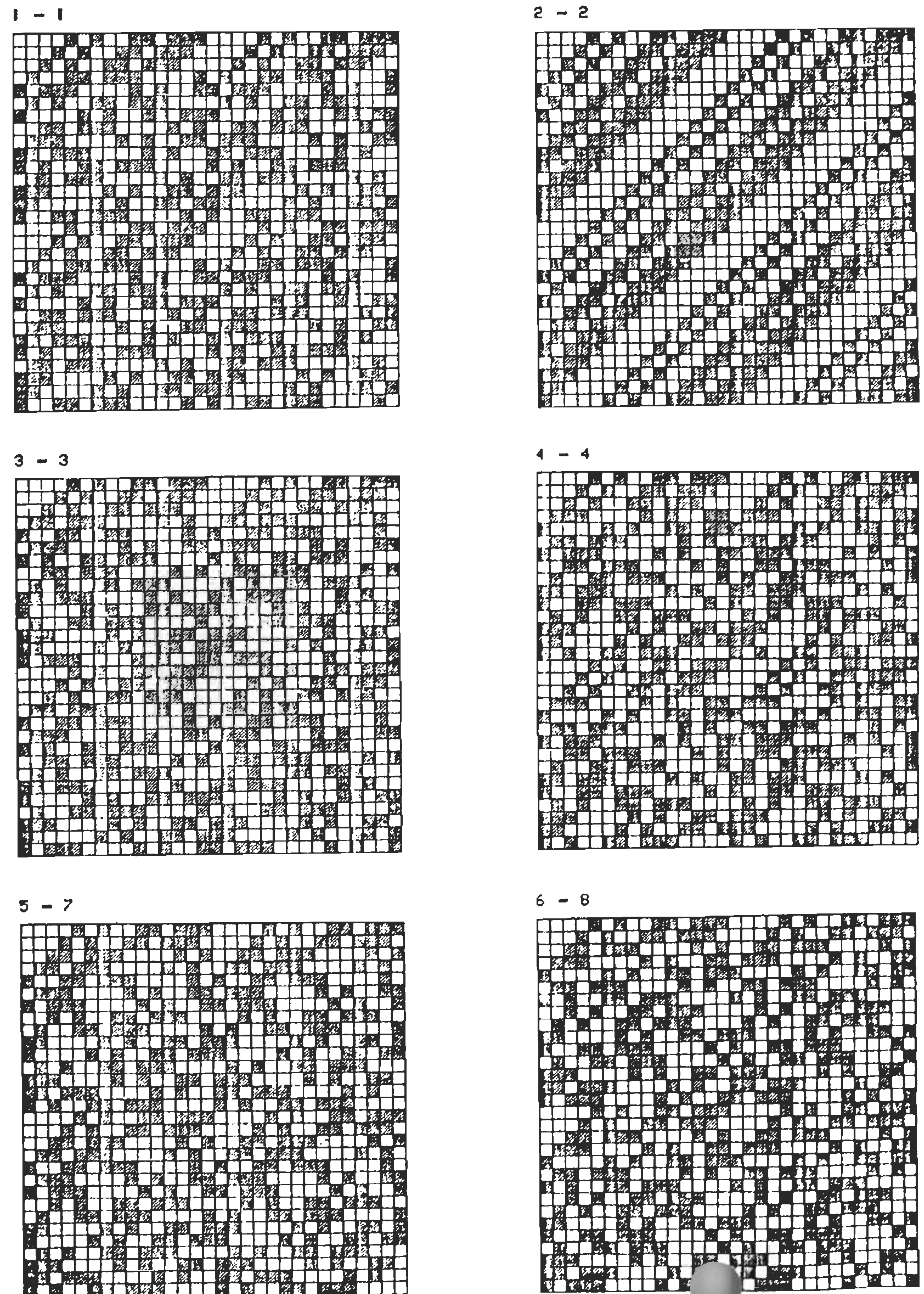


Figure 3. Sequential arrays with sequence σ_{15}.

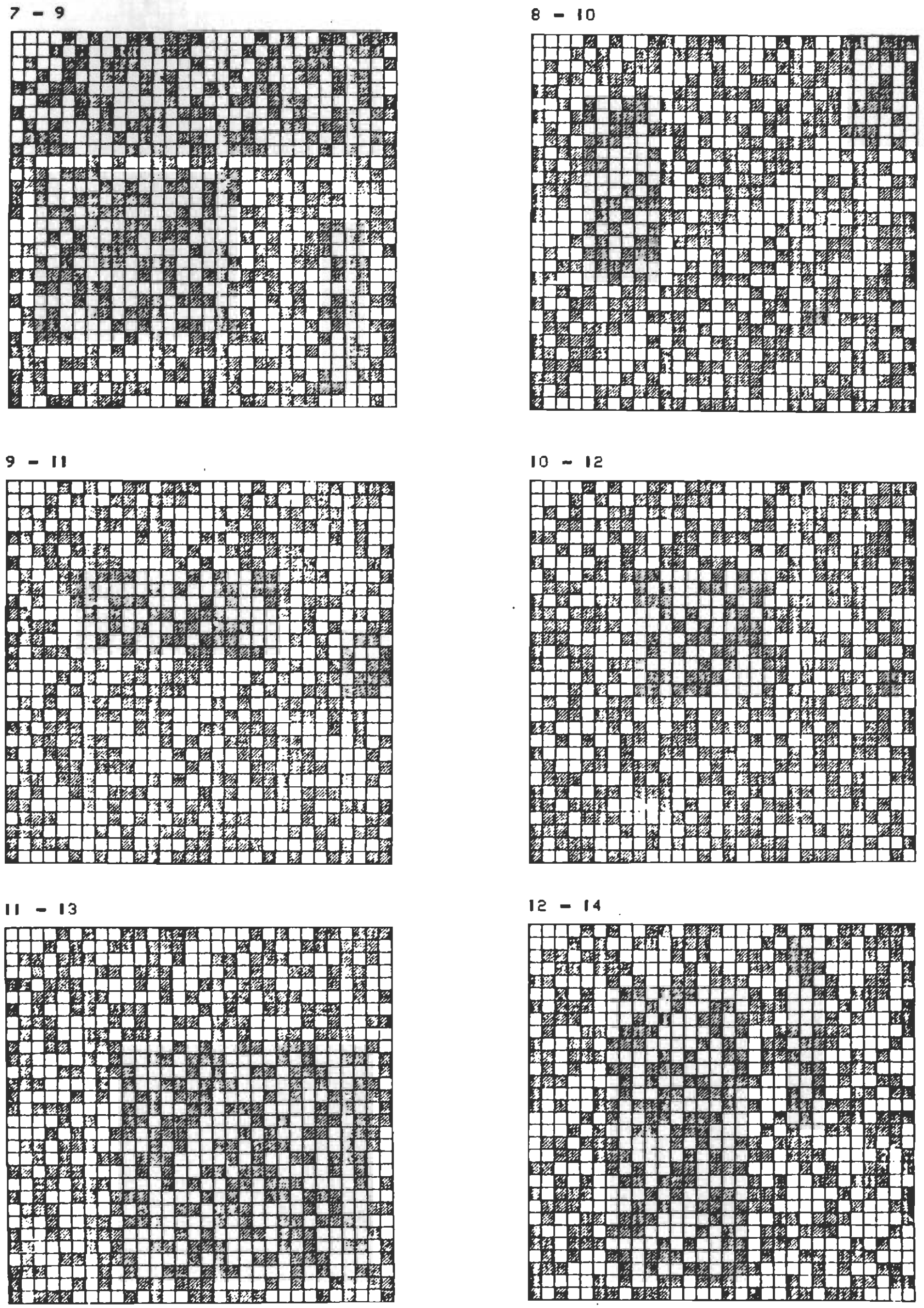


Figure 3. Sequential arrays with sequence δ_{15}.

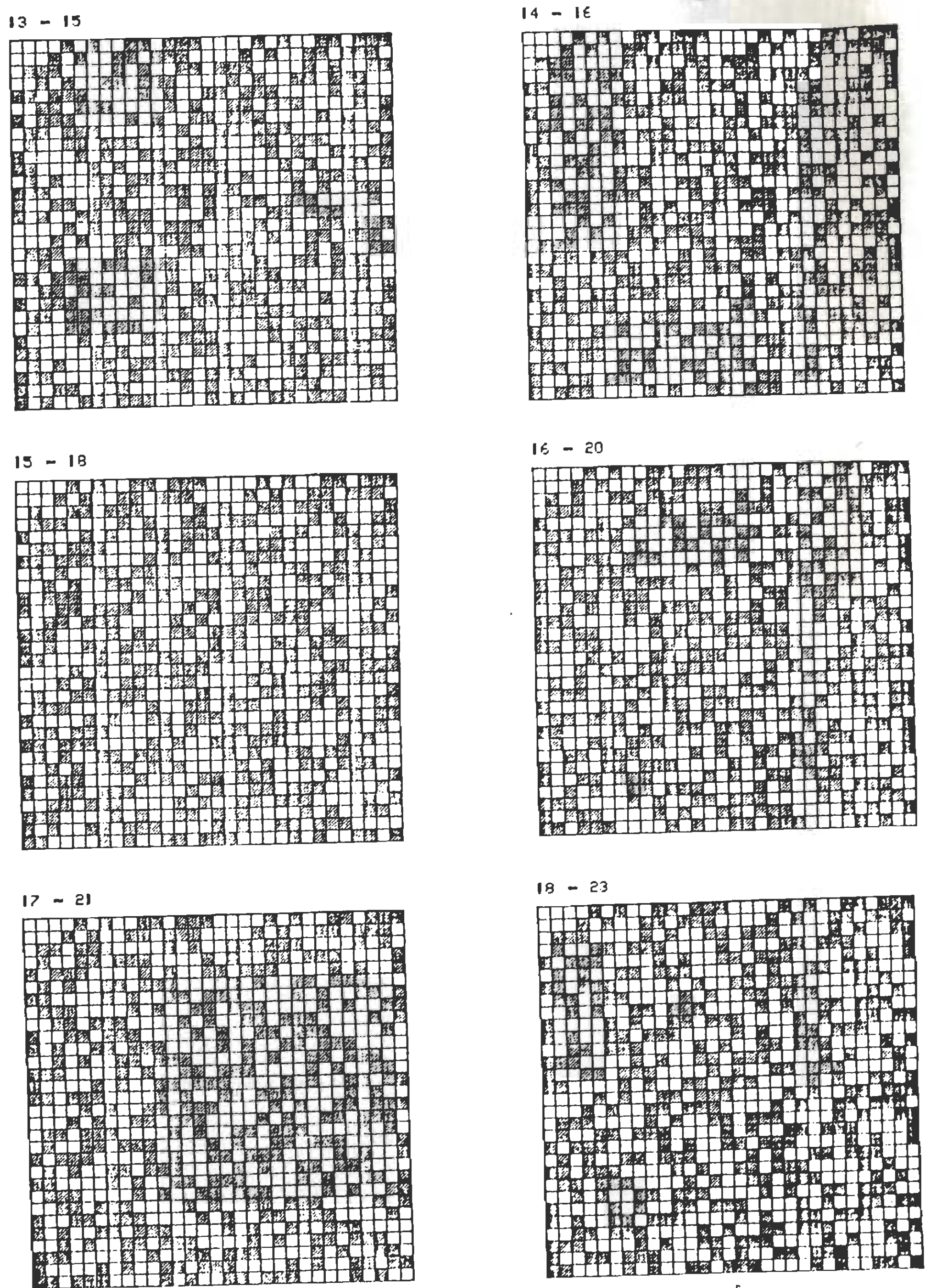
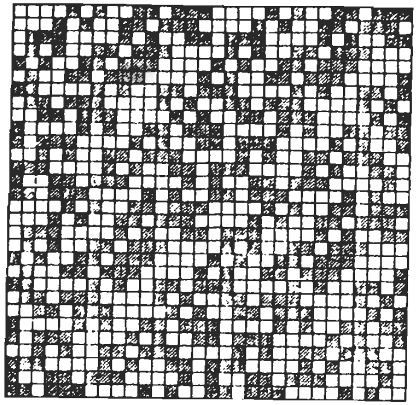
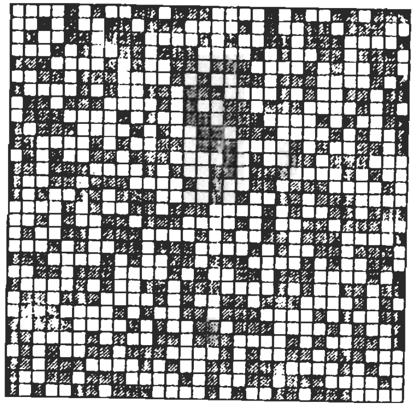


Figure 3. Sequential arrays with sequence δ_{15}.

19 – 24



20 – 33



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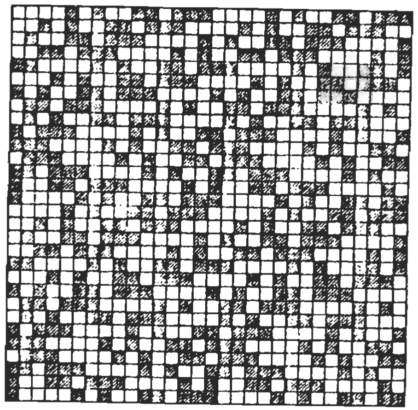

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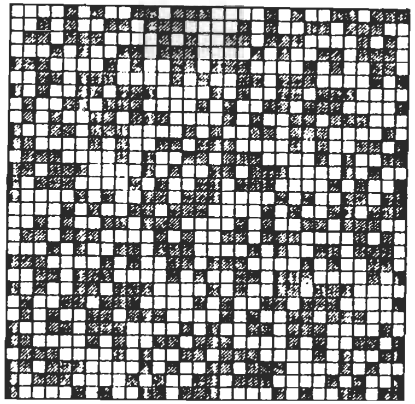
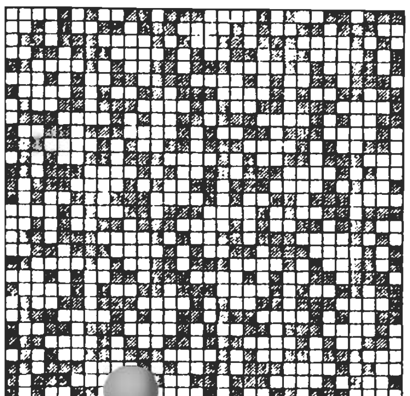


23 – 38



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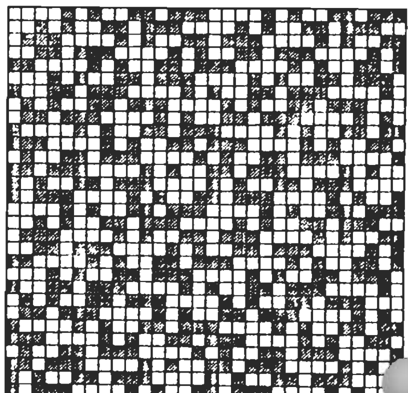
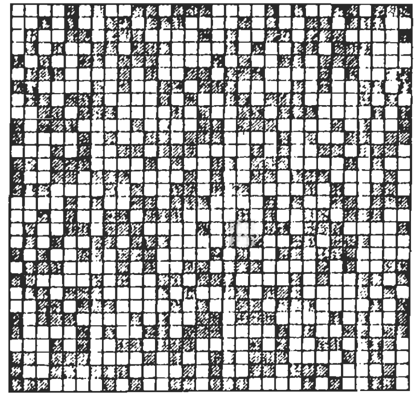


Figure 3. Sequential arrays with sequence δ_{15}.

25 – 41



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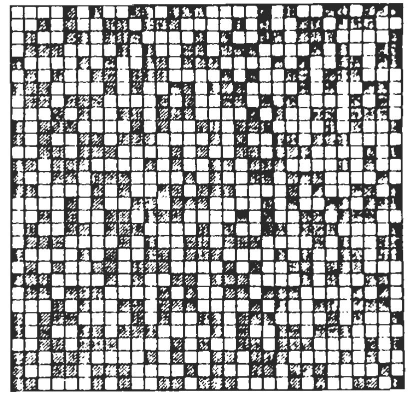

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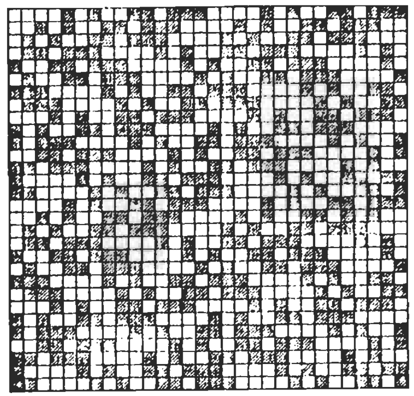

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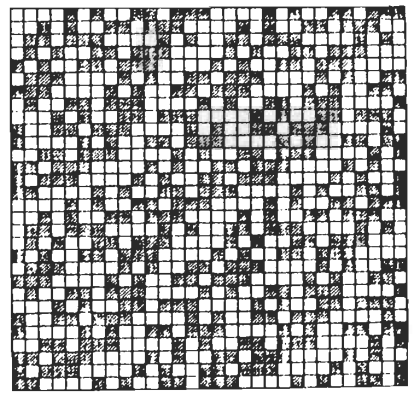

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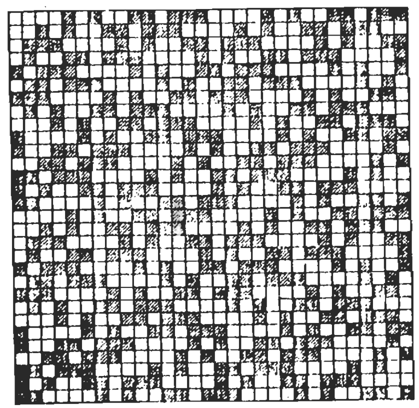

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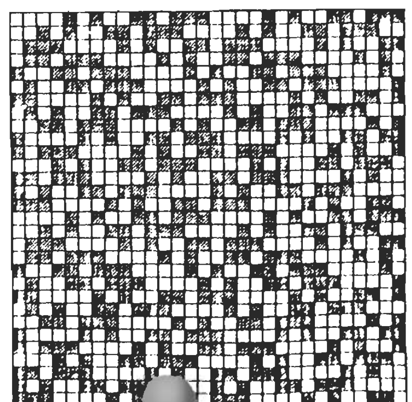
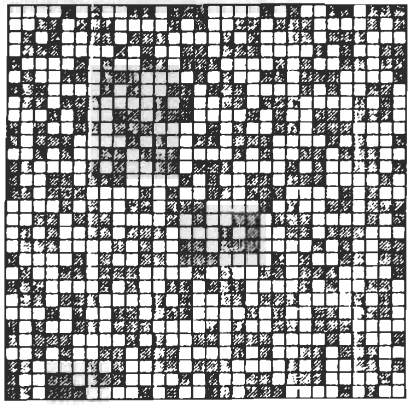


Figure 3. Sequential arrays with sequence δ_{15}.

31 - 68



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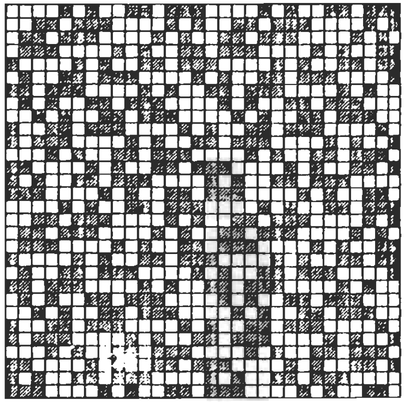

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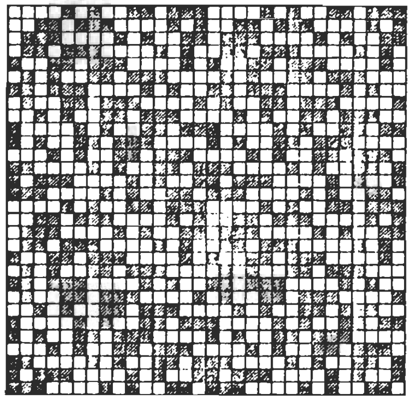

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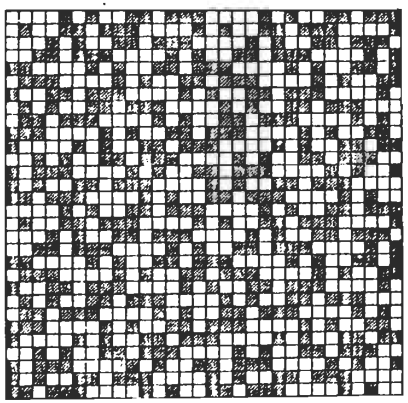

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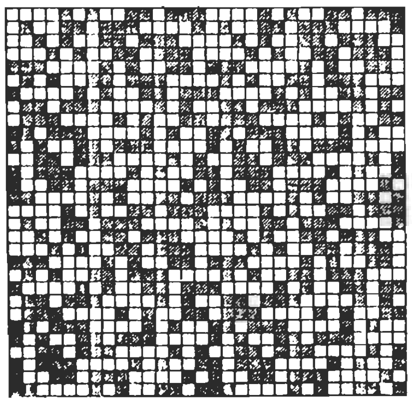

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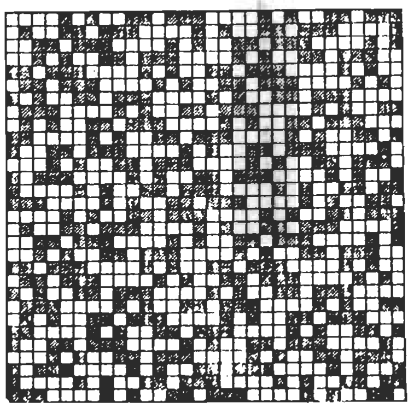
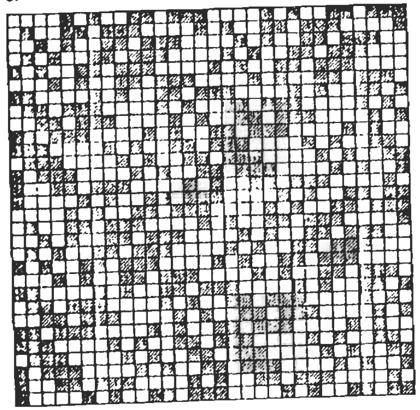


Figure 3. Sequential arrays with sequence δ_{15}.

37 - 81



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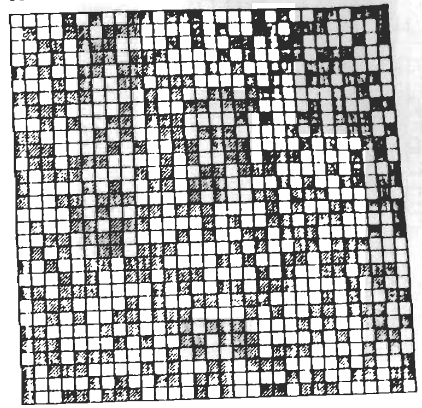

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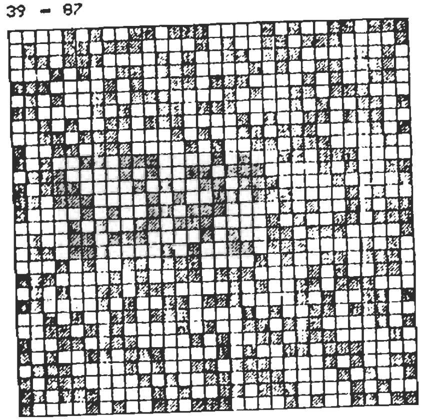

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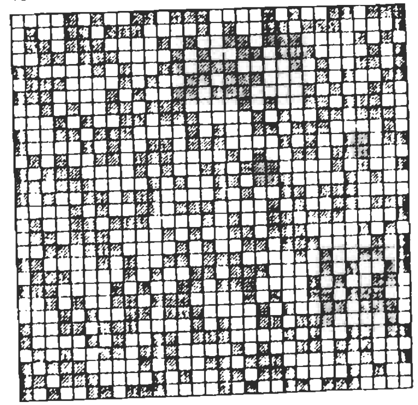

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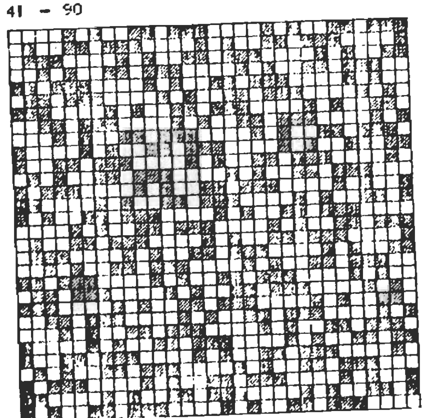


42 - 93

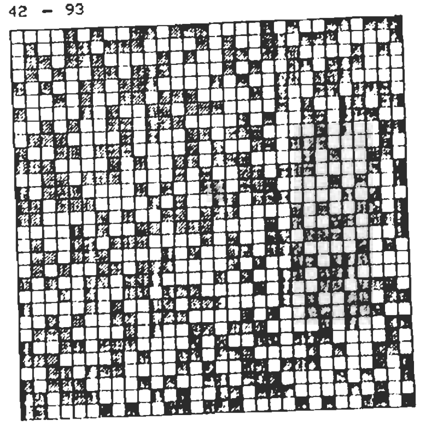
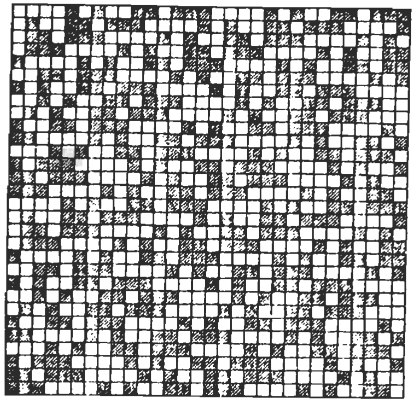


Figure 3. Sequential arrays with sequence δ_{15}.

43 - 94



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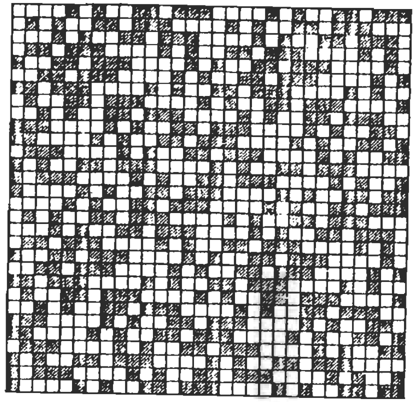

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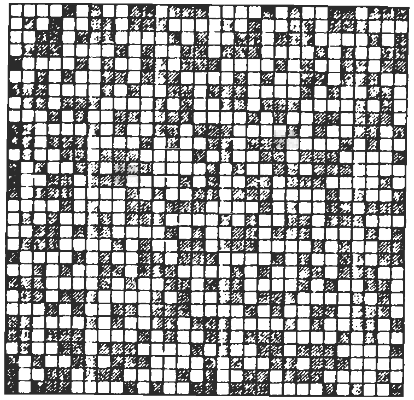
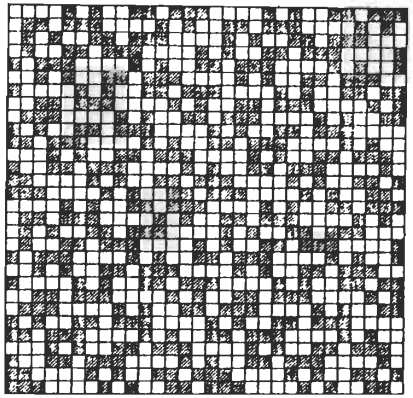


46 - 141



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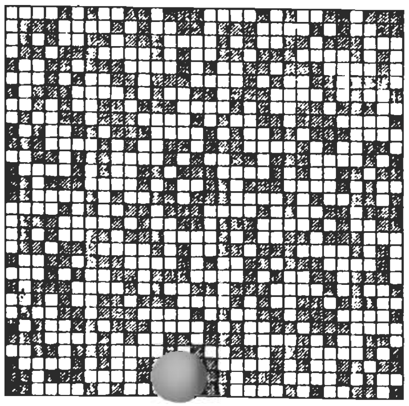

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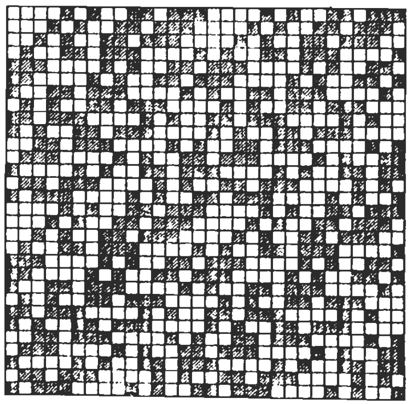
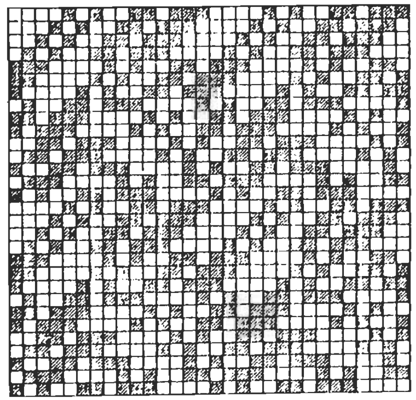


Figure 3. Sequential arrays with sequence δ_{15}.

49 - 170



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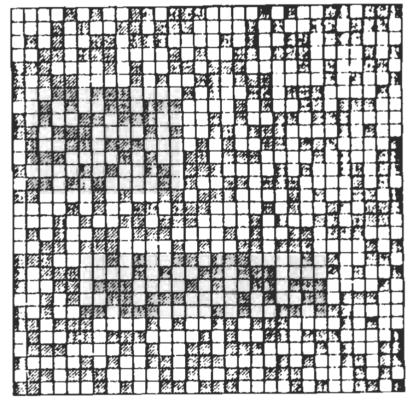

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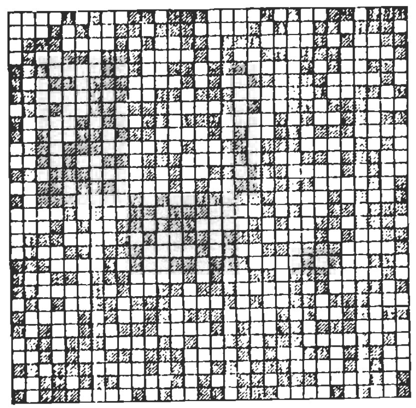

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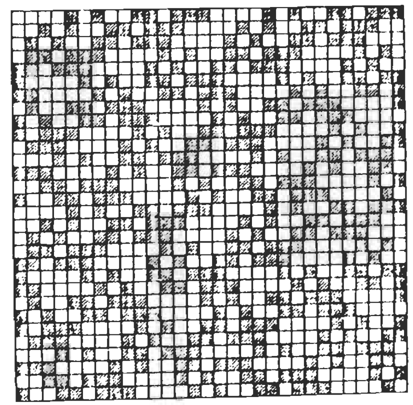

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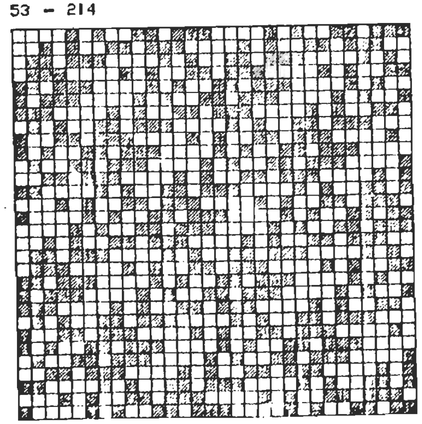

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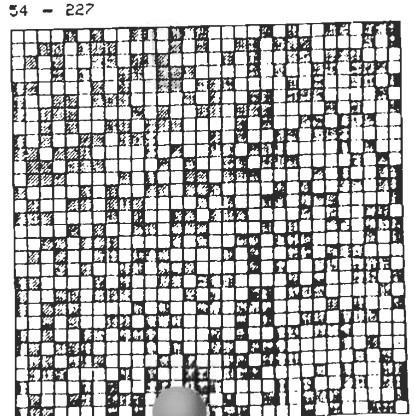
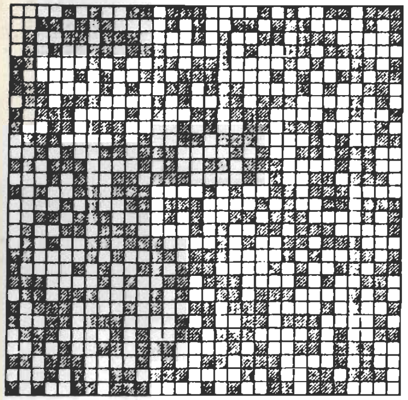


Figure 3. Sequential arrays with sequence δ_{15}.

55 - 233



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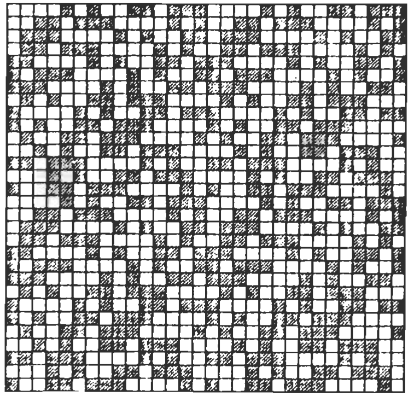

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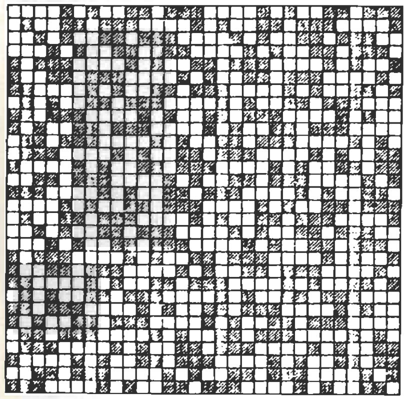
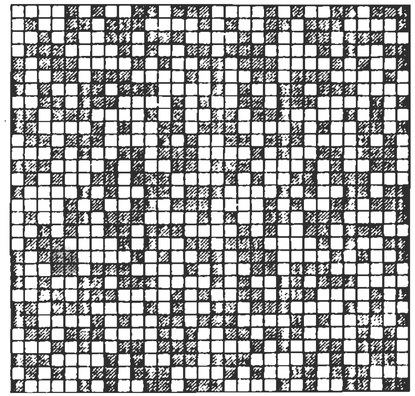


58 - 250



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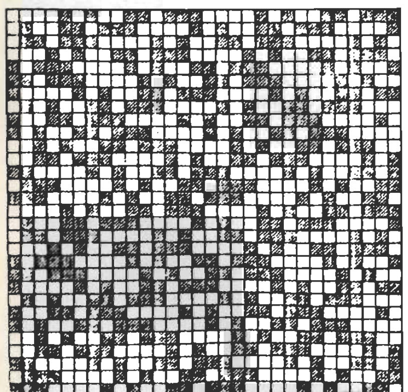

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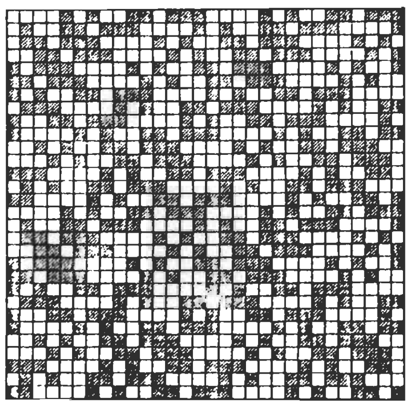
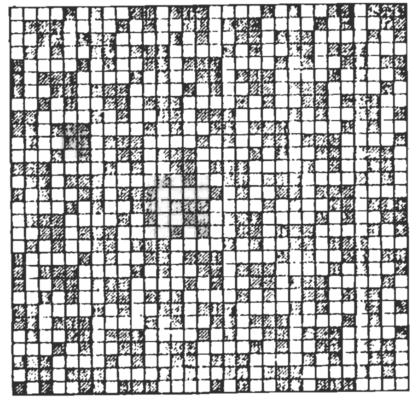


Figure 3. S[illegible]ntial arrays with sequence δ_{15}.

61 - 323



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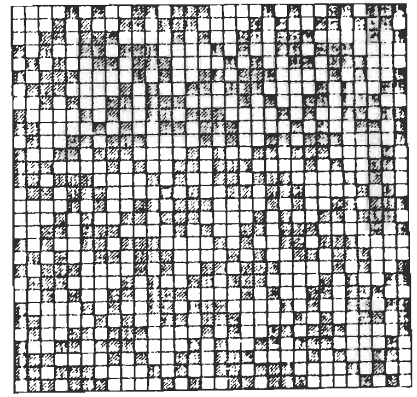

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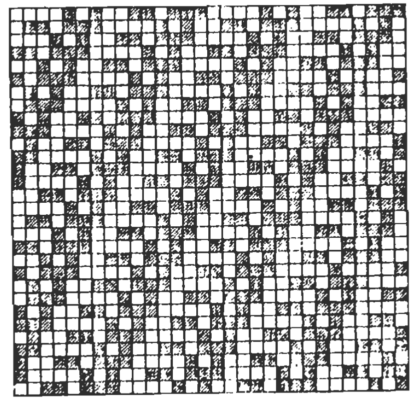

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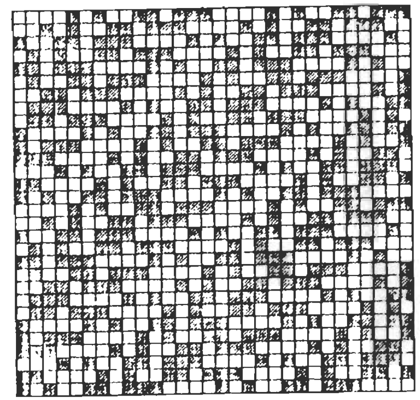

Figure 3. Sequential arrays with sequence δ_{15}.

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Department of Mathematics
University of Queensland
St. Lucia
Queensland

Department of Computer Science
University of Manitoba
Winnipeg
Manitoba
Canada

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Theorem.
$3-(q^2+1,q+1,1)$

Proof. Le
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$C_0, C_1, \ldots, C_{q-2}$
That is,

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developed by ad

$\{\infty, 0, 1, x^{q}$

$\{\infty, x^{(q+1)i+}$

$\{\infty, 0, x, x^{(}$

$\{\infty, x^{(q+1)j},$

$\{\infty, 0, x+x^{(}$

$\{\infty, x^{(q+1)j},$

Thus v =

and k = q+1.