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On enumerating certain design problems in terms of bicoloured graphs with no isolates

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Abstract. The enumeration of zero-one matrices with nonzero row and column sums has recently been solved in the context of certain problems arising in architectural design. However, this solution took no account of the symmetries of these configurations, which would be required for any realistic application in an architectural context. In graph-theoretic terms, this new problem calls for counting unlabelled bicoloured graphs with no isolated points. We determine these numbers by deriving their generating function. We also present an equivalent set of recurrence relations, from which the numbers may be easily computed.

1 Introduction

There are many design problems which involve a set of factors, V, and interrelationships between the factors, $\rho \subseteq V \times V$. Such design problems were first investigated in formal terms by Alexander and Manheim (1962) and Alexander (1964). Recently Batty (1974) has reexamined this class of problems in a Markovian decisionmaking context. It is usual to represent such relations by a $m \times m$ zero-one matrix, where $m = |V|$, or diagrammatically by means of a digraph of order m (Harary et al, 1965), as in the Batty paper.

Nevertheless many other design problems involve two sets of elements, U and W, and some relation, $\lambda \subseteq U \times W$, between them. Again we may represent this relation as a zero-one matrix, but in general the matrix will be rectangular of order $m \times n$, where $m = |U|$ and $n = |W|$. Alternatively, as we show below, an (m, n) bicoloured graph may be used. By way of illustration we refer briefly to four recent examples from the literature in which bicoloured graphs were or might have been used.

1.1 *Example 1 (Atkin, 1974a; 1974b; 1975)*

Many of Atkin's problems involve two sets. In one instance U is a set of architectural features, $\{u_1, u_2, ..., u_n\}$:

u_1 elaborate chimney;

u_2 Tudor overhang at first-floor level;

⋮

u_n large glass window;

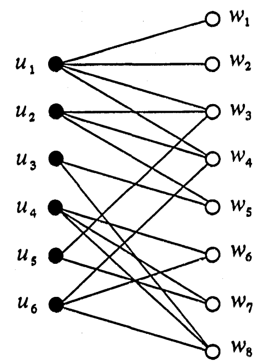
and W is a set of buildings, $\{w_1, w_2, ..., w_m\}$, where each building w_i is specified in terms of its feature set, a subset of 2^U, the power set of U.

Atkin (1974a, pages 60-61) illustrates his approach with an example using two sets U and W where $m = 6$ and $n = 8$. The incidence matrix for the relation $\lambda \subseteq U \times W$ is shown in figure 1. Atkin's mode of analysis goes on to represent this relation in terms of a simplicial complex, $K_W(U; \lambda)$, and its conjugate, $K_U(W; \lambda^{-1})$.

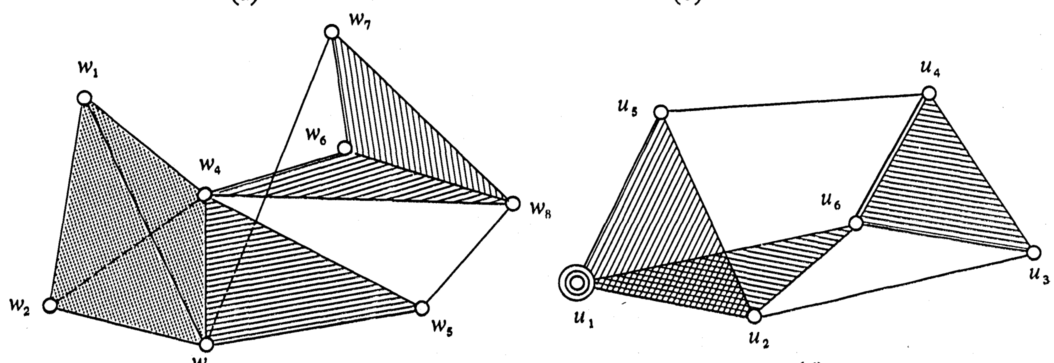
As Atkin points out, representations involving N-dimensional objects are difficult to illustrate for $N > 3$, and yet "in the realm of social science the *Euclidean spaces* which can accommodate the various mathematical relations will commonly be of much greater dimension than the 3-dimensional structure demanded by the physicist". In figure 1 we compare the two simplicial-complex diagrams of the matrix with the single bicoloured-graph representation. There are no dimensional constraints on the drawing of bicoloured graphs in the plane such as we find with multidimensional simplicial complexes, and the duality of $\lambda \subseteq U \times W$ and $\lambda^{-1} \subseteq W \times U$ is seen to be democratically preserved in the graph representation: it all depends on whether one looks from the set U across to W, or from W back across to U. Readers of Atkin's work will note that the degree $d(u_i) - 1$ is the dimension of the simplex u_i, and similarly for vertices in W. Other structural features described by Atkin in terms of simplicial complexes have their parallel in the terminology of bicoloured graphs. Indeed for many practitioners the language of graph theory could well be less forbidding than that of combinatorial topology.

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}
\begin{array}{c} \begin{array}{cccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

(a)



(b)



(c) (d)

Figure 1. The diagrammatic representations of Atkin's matrix (a) as: (b) a bicoloured graph; (c) the simplicial complex $K_W(U; \lambda)$; and (d) its conjugate $K_U(W; \lambda^{-1})$.

1.2 *Example 2 (March, 1976; Mitchell et al, 1976; Matela and O'Hare, 1976; Earl, 1977)*

In March (1976) it is proposed to use a *minimal representation* for patterns based on a rectangular grid. Since a grid is involved, bicoloured graphs may be used in representing such designs. The 'rows' of the grid form the set U, and the 'columns' of the grid constitute the set W. In most cases the minimality criteria for the representation imply that the bicoloured graph has no isolates (or vertices not incident to an edge).

Three types of *gridiron patterns* may be distinguished (figure 2): *2-patterns* in which the square *faces* of the grid form the elements of the design (for example, a set of city blocks in New York City); *1-patterns* in which the *lines* of the grid make up the elements of the design (for example, a set of streets and avenues between blocks in New York); and *0-patterns* in which the *points* of the grid contribute the design elements (for example, a set of street intersections in New York). 2-patterns and 0-patterns are in a mathematical sense dual concepts, but 1-patterns are different and this can be seen in the structure of the bicoloured graph representation. For an (m, n) grid, 2- and 0-patterns are representable as subgraphs of $K_{m,n}$, the complete (m, n) bicoloured graph, whereas 1-patterns are modelled by subgraphs of $K_{m,n-1} \cup K_{m-1,n}$.

An example of a 2-pattern is a *polyomino* discussed in an architectural context by Matela and O'Hare (1976), whereas a representative of a 1-pattern is the *rectangular dissection* introduced into design literature by Steadman (1973).

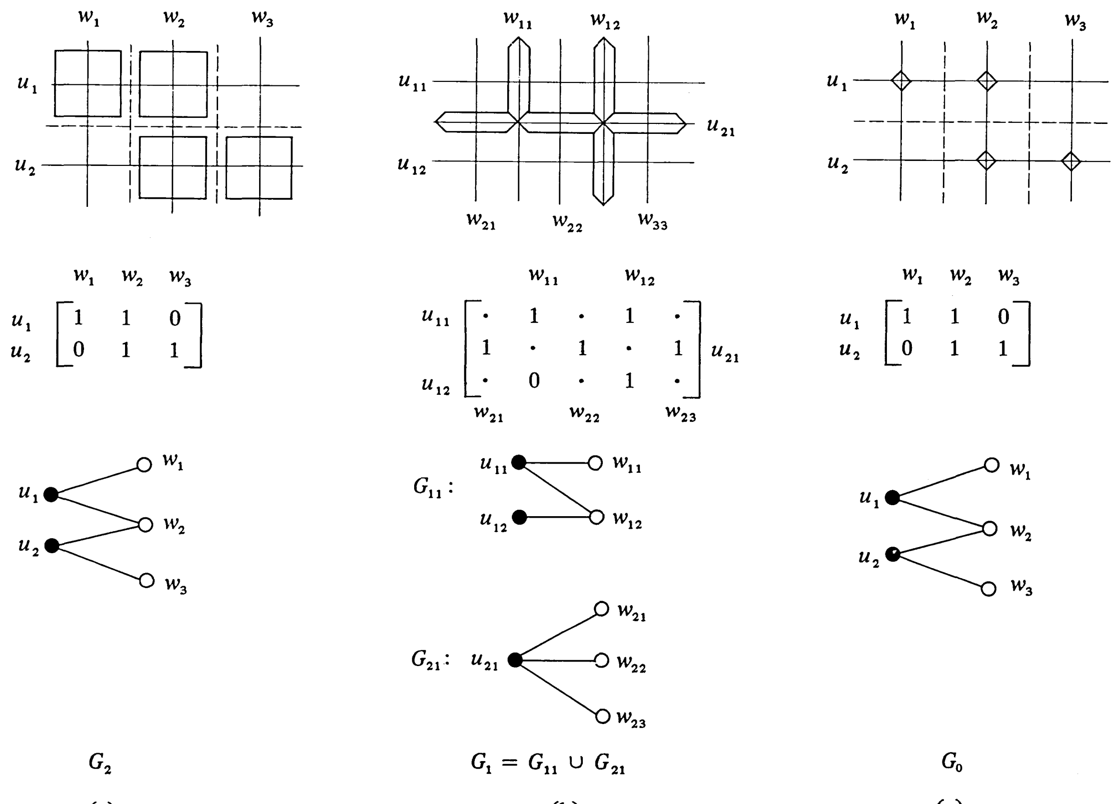


Figure 2. The representations of gridiron patterns on a (2, 3) grid in matrix and bicoloured-graph form: (a) a 2-pattern and its bicoloured graph $G_2 \subset K_{2,3}$; (b) a 1-pattern and its bicoloured graph $G_1 = G_{22} \cup G_{21}$, where $G_{11} \subset K_{2,2}$ and $G_{21} = K_{1,3}$; and (c) a 0-pattern and its graph $G_0 \subset K_{2,3}$.

1.3 *Example 3 (Fawcett, 1976a; 1976b; 1977)*

In his studies of adaptability in school buildings, Fawcett takes a set U of activities, $\{u_1, u_2, ..., u_n\}$:

u_1 demonstrations/practicals;

u_2 reading/writing;

$\vdots$

u_n physical experiment;

and a set W of space types, $\{w_1, w_2, ..., w_m\}$:

w_1 all-purpose laboratory;

w_2 light practical room;

$\vdots$

w_m seminar room.

He defines a relation $\lambda \subseteq U \times W$ showing which activities may feasibly take place in which types of space. The example we illustrate in figure 3 is taken from Fawcett (1976b).

$$\begin{array}{c|c} & \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

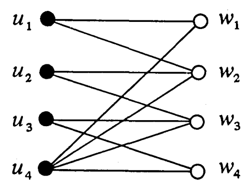
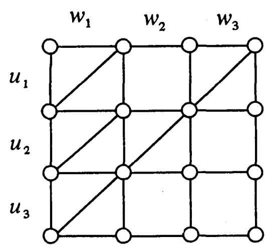


Figure 3. The diagrammatic representation of Fawcett's matrix as a (4, 4) bicoloured graph.

1.4 *Example 4 (Bolker and Crapo, 1977; Crapo, 1977)*
Recently it has been shown that minimal bracing schemes of a (m, n) rectangular framework correspond to spanning trees of the complete (m, n) bicoloured graph. Bolker and Crapo make U the set of east–west (E–W) halls in the framework and W the set of north–south (N–S) halls. The relation $\lambda \subseteq U \times W$ is an incidence relation indicating whether or not a particular bay of the framework (at the intersection of a E–W hall and a N–S hall) is cross-braced. Figure 4 shows an example from their joint paper (Bolker and Crapo, 1977, page 132).



$$\begin{array}{c|c} & \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

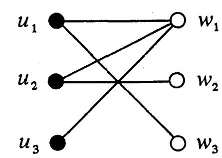


Figure 4. A rigid bracing scheme on a (3, 3) framework with its corresponding representations as a matrix and a bicoloured graph. The graph in this case is a spanning tree since the bracing is minimal.

1.5 *Discussion*
Widely different kinds of design problem may be represented by bicoloured graphs. The four examples indicate certain features of bicoloured graphs which are likely to interest the designer. In all four cases the bicoloured graphs do not have isolated vertices. This is likely to be a common feature of many design situations represented in this way. The reason for this is intuitively understandable. It would be unusual for a designer to consider elements which are unrelated to the problem he is studying. In Fawcett's case, every space that the designer provides will be expected to accommodate some activity, and every activity the designer is asked to consider will be provided for by a suitable room. This is a different situation from that faced by the school administrator who may allocate activities in such a way that some rooms may be left empty, or who is unlikely to plan for all possible activities to occur simultaneously in the timetable on each and every occasion. The administrator's task is to find a *matching* (Bondy and Murty, 1976) on the designer's bicoloured graph of feasible activity/space pairs. The designer who plans for adaptability will be interested in the set of all possible matchings from which the administrator might choose in the future. However, this is not our problem here.

In the case of cross-bracing a rectangular framework, it seems obvious that each hall must contain at least one brace if the structure is to be rigid and that the set of bicoloured graphs representing bracing schemes will be restricted to those with no isolates. So much for our interest in bicoloured graphs with no isolates. We now turn to the question of labelling.

Our first three problems clearly made use of labelled graphs, but returning to the problem concerning adaptibility in school buildings we find that Fawcett (1976b) comments that

> "We can take account of or ignore the distinctions of elements within types in four ways:
>
> | | | spaces non-distinct | spaces distinct |
> |---|---|---|---|
> | activities | non-distinct | I | II |
> | | distinct | III | IV |
>
> The actual measure of adaptability that we arrive at for given activity and spatial schedules will vary depending on which way we decide this issue ...".

The issue, in other words, of labelling or not labelling is of some consequence in design problems such as those addressed by Fawcett, but in the portrayal of gridiron patterns it is clearly important that the bicoloured graph is labelled since the labels correspond to the geometrical coordinates within the grid. However, equivalent gridiron patterns may be obtained through the symmetry operations of rotation and reflection on the rectangle or square. This equivalence condition requires that the labels of the two sets U and W are permuted as follows:

U: $(1\ m)\ (2\ m-1)$ ... and W: $(1\ n)\ (2\ n-1)$ ... ,

and when $m = n$ we also have the permutation

U, W: $(u_1\ w_1)(u_2\ w_2)$ ...

in which the sets U and W are interchanged.

Most geometrical design problems are unlikely to show equivalence under complete permutation of the rows and columns of the grid; that is, under the action of the symmetric groups S_m on U and S_n on W. However, this turns out to be true of the bracing schemes studied by Bolker and Crapo. These authors have shown that cross-braced rectangular frameworks are structurally equivalent under such permutations. It is therefore useful in this and possibly other design contexts to enumerate *unlabelled* bicoloured graphs.

We will demonstrate the equivalence between binary (zero-one) matrices and bicoloured graphs. The matrix representation is algebraic, the graph-theoretic depiction is structural, and the two are fully interchangeable. The graphical viewpoint lends itself admirably to the handling of structural symmetry by means of the automorphism group of a graph. The enumeration question at hand is then the determination of the number of unlabelled bicoloured graphs with no isolated points. We shall see that the most difficult aspect of this problem has already been solved, as the total number of bicoloured graphs has been found. Building on this previous result, we derive the desired formulae. In addition we shall present tables of numerical values.

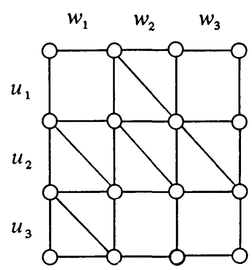
All technical terms not explained here can be found in Harary and Palmer (1973).

2 Binary matrices versus bicoloured graphs

Our examples in section 1 have served to illustrate the correspondence between the binary (zero-one) matrices with positive row and column sums studied by Jackson (1976) and bicoloured graphs with no isolated points.

An (m, n) bicoloured graph G has m umber points in the set U and n white points in the set W, with two points adjacent only if one is umber and the other is white. The construction of the $m \times n$ adjacency matrix $\mathbf{A}$ of a given bicoloured graph G whose points are coloured umber, $U = \{u_1, u_2, ..., u_m\}$, or white, $W = \{w_1, w_2, ..., w_n\}$, is given at once by setting $a_{ij} = 1$ whenever u_i and w_j are adjacent and $a_{ij} = 0$ otherwise. Obviously this gives a one-to-one correspondence between bicoloured graphs and binary matrices. The ith row sum and the jth column sum of matrix $\mathbf{A}$ give the degrees of the points u_i and w_j. By definition an *isolate*, or *isolated point*, is not incident with any lines. Hence a binary matrix has positive row and column sums if and only if its bicoloured graph has no isolated points.

When the labels $u_1, u_2, ..., u_m$ of the umber points and $w_1, w_2, ..., w_n$ of the white points are fixed, the result is called a *labelled bicoloured graph*. However, the structure of a graph is determined by its isomorphism class rather than its particular labelling. For example, we show in figure 5 another labelling of the spanning tree shown in figure 4 and its corresponding adjacency matrix.



$$\begin{array}{cc} & \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

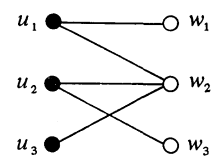


Figure 5. Another rigid bracing of the framework shown in figure 4. This is equivalent to another labelling of the bicoloured graph shown in that figure under the permutations $(u_1 u_2)(u_3)$ and $(w_1 w_2)(w_3)$.

3 Bicoloured graphs

Let $b_{m,n}$ be the number of bicoloured graphs with m umber points and n white points. The perusal of figure 6 verifies that $b_{3,2} = 13$, since all the (3, 2) bicoloured graphs are displayed. In this figure we use the mnemonic convention that the solid points are umber and the open points are white.

To determine a general formula for $b_{m,n}$, we require the symmetric group S_m in order to take into account the interchangeability amongst themselves of the umber

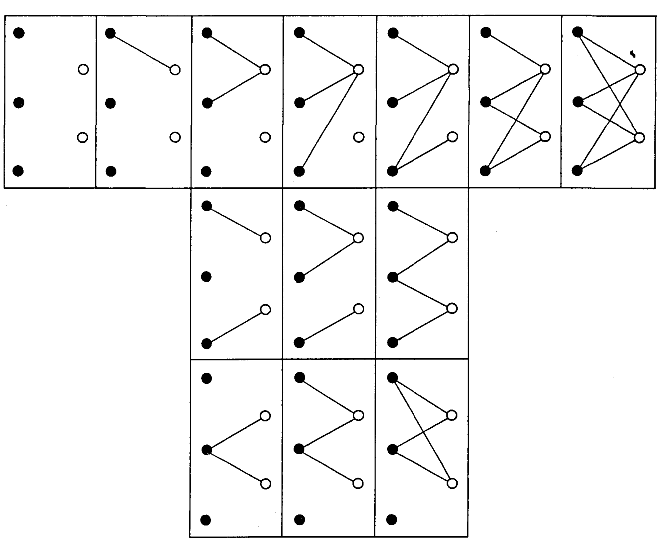


Figure 6. The (3, 2) bicoloured graphs.

points, and similarly we need S_n for the white points. The combined action of these two groups on the mn possible lines joining U and W is given by their cartesian product $S_m \times S_n$. In accordance with Pólya's classical enumeration theorem, $b_{m,n}$ is obtained by substituting 2 for each of the variables in the cycle index $Z(S_m \times S_n)$, which can be written by use of standard notation as the equation

$$b_{m,n} = Z(S_m \times S_n, 2) \ . \tag{1}$$

As shown in Harary (1958), it follows easily that $b_{m,n}$ can be written explicitly in the form

$$b_{m,n} = \sum{}^{*} 2^{\sum_{i,j} r_i s_j (i,j)} \Big/ \prod_i r_i! s_i! i^{r_i + s_i} \ . \tag{2}$$

In this equation (i, j) denotes the greatest common divisor of i and j. The asterisk on the summation sign indicates both in equation (2) and in later equations that the sum is taken over all pairs of sequences $r_1, ..., r_m$ and $s_1, ..., s_n$ of nonnegative integers satisfying $m = \Sigma i r_i$ and $n = \Sigma i s_i$. We denote by $b_{m,n,q}$ the number of (m, n) bicoloured graphs with q lines. As shown in figure 6, $b_{3,2,4} = 3$. For fixed m and n these numbers are conveniently coded by the ordinary generating function defined by

$$b_{m,n}(z) = \sum_{q=0}^{mn} b_{m,n,q} z^q \ . \tag{3}$$

By Pólya's enumeration theorem we obtain a formula for $b_{m,n}(z)$ which is similar to equation (1), namely

$$b_{m,n}(z) = Z(S_m \times S_n, 1+z) \ . \tag{4}$$

Using the known formula for $Z(S_m \times S_n)$, we can write an explicit formula for this generating function:

$$b_{m,n}(z) = \sum{}^{*} \prod_{i,j} (1+z^{[i,j]})^{r_i s_j (i,j)} \Big/ \prod_i r_i! s_i! i^{r_i + s_i} \ . \tag{5}$$

Here $[i, j]$ denotes the least common multiple of i and j, and (i, j) is the greatest common divisor.

Equations (2) and (5) are most suitable for the computation of numerical values of $b_{m,n}$ and $b_{m,n,q}$. These numbers will be useful in achieving our goal of counting bicoloured graphs without isolates, as will be seen in the next section.

4 Bicoloured graphs with no isolated points

Obviously the total number of labelled bicoloured graphs with m umber points and n white points, including those with isolates, is 2^{mn}. Jackson (1976) obtained a formula for the number of labelled bicoloured graphs with no isolates, having a given number, q, of lines.

Clearly $b_{m,n}$ is the total number of (m, n) bicoloured graphs, including those with isolates—see figure 6. We now derive the number $b'_{m,n}$ of unlabelled (m, n) bicoloured graphs with no isolated points, and the number $b'_{m,n,q}$ of such graphs having q lines. This is most conveniently done in terms of the generating functions $b(x, y)$ and $b'(x, y)$ for all bicoloured graphs and for those with no isolates, respectively, which are defined by

$$b(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} x^m y^n \ , \tag{6}$$

under the convention that $b_{m,n} = 1$ whenever $m = 0$ or $n = 0$, and

$$b'(x, y) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b'_{m,n} x^m y^n \ . \tag{7}$$

Table 1. The number $b_{m,n,q}$ of (m, n) bicoloured graphs with q lines for $2 \leqslant m+n \leqslant 10$ and gives the total number $b_{m,n}$ of (m, n) bicoloured graphs.

q	$m+n=2$	$m+n=3$	$m+n=4$		$m+n=5$		$m+n=6$			$m+n=7$		
	(1, 1)	(2, 1)	(3, 1)	(2, 2)	(4, 1)	(3, 2)	(5, 1)	(4, 2)	(3, 3)	(6, 1)	(5, 2)	(4, 3)
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	1	3	1	3	1	3	3	1	3	3
3			1	1	1	3	1	3	6	1	3	6
4				1	1	3	1	6	7	1	6	11
5						1	1	3	7	1	6	13
6						1		3	6	1	6	17
7								1	3		3	13
8								1	1		3	11
9									1		1	6
10											1	3
11												1
12												1
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25												
$b_{m,n}$	2	3	4	7	5	13	6	22	36	7	34	87

Table 2. The number $b'_{m,n,q}$ of (m,n) bicoloured graphs with q lines and no isolates for This table also gives the total number $b'_{m,n}$ of (m,n) bicoloured graphs without isolates.

q	$m+n=2$	$m+n=3$	$m+n=4$		$m+n=5$		$m+n=6$			$m+n=7$		
	(1, 1)	(2, 1)	(3, 1)	(2, 2)	(4, 1)	(3, 2)	(5, 1)	(4, 2)	(3, 3)	(6, 1)	(5, 2)	(4, 3)
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0
2		1	0	1	0	0	0	0	0	0	0	0
3			1	1	0	1	0	0	1	0	0	0
4				1	1	2	0	2	2	0	0	1
5						1	1	2	5	0	2	4
6						1		2	4	1	3	9
7								1	3		2	9
8								1	1		2	9
9									1		1	5
10											1	3
11												1
12												1
13												
14												
15												
16												
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23												
24												
25												
$b'_{m,n}$	1	1	1	3	1	5	1	8	17	1	11	42

$m \geqslant n$. Note that for $n = 0$ there is only one $(m, 0)$ bicoloured graph for all m. This table also

$m+n=8$				$m+n=9$				$m+n=10$					q
(7, 1)	(6, 2)	(5, 3)	(4, 4)	(8, 1)	(7, 2)	(6, 3)	(5, 4)	(9, 1)	(8, 2)	(7, 3)	(6, 4)	(5, 5)	
1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	3	3	3	1	3	3	3	1	3	3	3	3	2
1	3	6	6	1	3	6	6	1	3	6	6	6	3
1	6	11	16	1	6	11	16	1	6	11	16	16	4
1	6	18	21	1	6	18	27	1	6	18	27	34	5
1	10	26	39	1	10	33	53	1	10	33	62	69	6
1	6	29	44	1	10	41	80	1	10	49	100	130	7
	6	29	55	1	10	52	120	1	15	68	180	234	8
	3	26	44		6	54	140	1	10	85	250	367	9
	3	18	39		6	52	159		10	92	353	527	10
	1	11	21		3	41	140		6	92	400	669	11
	1	6	16		3	33	120		6	85	452	755	12
		3	6		1	18	80		3	68	400	755	13
		1	3		1	11	53		3	49	353	669	14
		1	1			6	27		1	33	250	527	15
			1			3	16		1	18	180	367	16
						1	6			11	100	234	17
						1	3			6	62	130	18
							1			3	27	69	19
							1			1	16	34	20
										1	6	16	21
											3	6	22
											1	3	23
											1	1	24
												1	25
8	50	190	317	9	70	386	1053	10	95	734	3250	5624	$b_{m,n}$

$2 \leqslant m+n \leqslant 10$ and $m \geqslant n$. Note that for $n = 0$ there is only one $(m, 0)$ bicoloured graph for all m.

$m+n=8$				$m+n=9$				$m+n=10$					q
(7, 1)	(6, 2)	(5, 3)	(4, 4)	(8, 1)	(7, 2)	(6, 3)	(5, 4)	(9, 1)	(8, 2)	(7, 3)	(6, 4)	(5, 5)	
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	3
0	0	0	1	0	0	0	0	0	0	0	0	0	4
0	0	2	2	0	0	0	1	0	0	0	0	1	5
0	3	6	11	0	0	3	5	0	0	0	2	2	6
1	3	14	21	0	3	9	20	0	0	4	8	14	7
	3	16	34	1	4	20	47	0	4	12	37	49	8
	2	19	33		3	26	76	1	4	28	82	131	9
	2	14	33		3	32	105		4	37	160	248	10
	1	10	19		2	29	109		3	49	230	410	11
	1	5	14		2	26	99		3	50	305	531	12
		3	6		1	15	71		2	49	305	601	13
		1	3		1	10	49		2	37	290	566	14
		1	1			5	25		1	27	218	474	15
			1			3	15		1	15	161	336	16
						1	6			10	93	222	17
						1	3			5	58	124	18
							1			3	26	67	19
							1			1	15	32	20
										1	6	16	21
											3	6	22
											1	3	23
											1	1	24
												1	25
1	15	91	179	1	19	180	633	1	24	328	2001	3835	$b'_{m,n}$

The function $b(x, y)$ of equation (6) is known because its coefficients are given in equation (2). The formalism in equation (7) of introducing the term 1 on the right-hand side leads to a neater statement of the results; intuitively it says that the 'empty graph' has no isolated points (Harary and Read, 1974). The summations can then start at $m = 1$ and $n = 1$ since a nonempty bicoloured graph with no isolates must have at least one umber point and at least one white point.

The relation between $b'(x, y)$ and $b(x, y)$ which we now derive is based on the following observation: an arbitrary bicoloured graph consists of the subgraph induced by all of its nonisolated points, together with the required number of isolated umber and white points. In terms of generating function this can be expressed by

$$b(x,y) = b'(x,y)(1+x+x^2+...)(1+y+y^2+...) . \tag{8}$$

Here the factor $1+x+x^2+...$ allows for adding 0, 1, 2, ... isolated umber points, whereas $1+y+y^2+...$ allows independently for adding 0, 1, 2, ... isolated white points.

Relation (8) can now be solved for $b'(x, y)$ by multiplying through by $(1-x)(1-y)$, to give

$$b'(x,y) = b(x,y)(1-x)(1-y) . \tag{9}$$

By equating coefficients of $x^m y^n$ on both sides of equation (9), one sees at once that this is equivalent to the recurrence relation

$$b'_{m,n} = b_{m,n} - b_{m-1,n} - b_{m,n-1} + b_{m-1,n-1} . \tag{10}$$

This recurrence holds for all $m, n \geqslant 0$ provided one understands the value of $b_{i,j}$ to be zero if i or j is negative. It is of course recurrence relation (10) rather than generating-function expression (9) that lends itself to the most direct numerical implementation by computer.

If it is desired to compute $b'_{m,n,q}$ then the appropriate generating functions to start from are defined by

$$b(x,y,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{mn} b_{m,n,q} x^m y^n z^q , \tag{11}$$

under the convention that $b_{m,n,0} = 1$ whenever $m = 0$ or $n = 0$, and

$$b'(x,y,z) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{mn} b'_{m,n,q} x^m y^n z^q . \tag{12}$$

Again the function $b(x, y, z)$ is known because its coefficients can be found from equation (5).

The relationship between $b(x, y, z)$ and $b'(x, y, z)$ is exactly parallel to that between $b(x, y)$ and $b'(x, y)$. The reason for this is that adding an isolated point to a graph requires no additional lines. Thus by analogy with equation (9) we may write

$$b'(x,y,z) = b(x,y,z)(1-x)(1-y) . \tag{13}$$

Equation (13) is equivalent to theorem 1 of Harary and Prins (1963), and equation (9) follows directly by setting $z = 1$. Equating the coefficients of $x^m y^n z^q$ on both sides of equation (13), we obtain the recurrence relation

$$b'_{m,n,q} = b_{m,n,q} - b_{m-1,n,q} - b_{m,n-1,q} + b_{m-1,n-1,q} . \tag{14}$$

This is valid for all $m, n, q \geqslant 0$ provided that $b_{i,j,k}$ is understood to be zero if i or j should be negative.

Recurrence relations (10) and (14) are eminently suitable for machine computation. In tables 1 and 2 are presented selected results obtained by computer for the numbers of unlabelled bicoloured graphs, with and without isolated points.

5 Further developments

The problems posed by Fawcett (example 3) in which one set of nodes in a bicoloured graph may be labelled and the other set unlabelled may be enumerated by use of $E_m \times S_n$ and $S_m \times E_n$ as required, where E is the identity group of appropriate order. The numbers without isolates could be easily obtained by use of a combination of the methods for deriving equations (9) and (13) with the methods of Jackson (1976).

When counting (m, n) bicoloured graphs here, it has been assumed that when $m = n$ the colours are not interchangeable. This makes practical sense when the two sets U and W refer to different collections of objects (for example, features and buildings, activities and spaces, and so on). However, in the case of gridiron patterns and cross-braced rectangular frameworks it is clearly possible to exchange E–W halls and N–S halls when their numbers are the same. The number $b_n(x)$ that enumerates bicoloured graphs with n points of each colour is given by Harary (1958):

$$b_n(x) = Z([S_n]^{S_2}, 1+x) , \tag{15}$$

where $[S_n]^{S_2}$ is the *exponentiation* of S_n with S_2. Formulae for $b_{n,q}$, b'_n, and $b'_{n,q}$ similar to those for the general (m, n) bicoloured graph are derived by the methods used above (tables 3 and 4). Details are given in Harary (1969) and Harary and Palmer (1973).

It is also possible to enumerate the number of *connected* (m, n) bicoloured graphs (Harary and Prins, 1963)—a problem equivalent to counting the number of bicolourable

Table 3. The number $b_{n,q}$ of (n, n) bicoloured graphs with q lines in which the colours are interchangeable, together with the total number of such graphs, b_n.

q	n = 2	3	4	5
0	1	1	1	1
1	1	1	1	1
2	2	2	2	2
3	1	4	4	4
4	1	5	10	10
5		5	13	20
6		4	23	39
7		2	26	72
8		1	32	128
9		1	26	198
10			23	280
11			13	353
12			10	399
13			4	399
14			2	353
15			1	280
16			1	198
17				128
18				72
19				39
20				20
21				10
22				4
23				2
24				1
25				1
b_n	6	26	192	3014

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Table 4. The number $b'_{n,q}$ of (n, n) bicoloured graphs with q lines and no isolates in which the colours are interchangeable, together with the total number of such graphs, b'_n.

q	n = 2	3	4	5
0	0	0	0	0
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	2	1	0
5		4	2	1
6		3	8	2
7		2	14	10
8		1	21	31
9		1	20	76
10			20	137
11			12	221
12			9	285
13			4	321
14			2	301
15			1	253
16			1	182
17				122
18				69
19				38
20				19
21				10
22				4
23				2
24				1
25				1
b'_n	3	14	115	2086

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graphs (a *bicolourable graph* is one which can be bicoloured). It is well known in graph theory that a given graph G can be bicoloured if and only if G has no cycles of odd length. The number of connected bicoloured graphs of order p equals the number of connected bicolourable graphs of order p. If $c_{m,n}$ is the number of connected (m, n) bicoloured graphs and $c_{m,n,q}$ is the number with q lines then, when $q = m+n-1$, $c_{m,n,q} = t_{m,n}$, the number of (m, n) bicoloured trees. It is this number $t_{m,n}$ that enumerates the bracing schemes of a (m, n) rectangular framework.

A three-coloured graph may be used to describe a three-dimensional rectangular framework in an analogous manner to the bicoloured graph used to represent a two-dimensional grid. The uses of such a representation in architectural work are likely to require *labelling*. The number of k-coloured graphs on labelled nodes was reported in Read (1960). Without labels it is worth noting that, more generally, Robinson (1968) devised a formula for counting k-coloured graphs in which the colours are interchangeable. Such graphs without isolates can also be enumerated.

Labelled k-coloured graphs have recently been studied in the context of engineering analysis (Onadera, 1973; Onadera et al, 1977) since they represent those branches of a network which 'tear' it into k-component subnetworks according to the diakoptical methods of Kron (1963). 'Tearing' a complex engineering system in this way corresponds to Alexander's (1964) aim of decomposing an architectural design problem into a set of subproblems. The enumeration of distinct, nonisomorphic structures (unlabelled k-coloured graphs) may therefore be of some interest to design theorists examining diakoptical or decomposition techniques.

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