Scan A 7240
etc

M Somoz

Emails, 7 pages

add to 3 (I think)

saas

Say

A7191
A7267
A7240

From alpha.ces.cwru.edu!somos Sun Jul 25 14:39:32 0400 1993
From: Michael Somos <somos@alpha.ces.cwru.edu>

Dear Neil,

In my research I have come across a few q-series of which I only know the first few terms. I wonder if you could identify them. They may even be related to modular forms, but I am not sure how to check.

```
f1= {1, 7, 15, 71, 106, 273, 486, 961, 1563, 3040}
f2= {1, 13, -45, 748, -10359, 169380, -2945617, 53668795,
    -1010587698, 19510685605, -384118017962, 7682318412651}
f3= {1,15,-241,13712,-792287,47727408,-2991954241}
f4= {1,31,-2848,413823,-68767135,12310047967,-2309368876639,
    447436508910495,-88755684988520798,17924937024841839390,
    -3671642907594608226078,760722183234128461061246}
f5= {1,47,-713,37847,-2000310,121185241,-7837254234,530465339249}
```

A — = A58206
B = ?
C = ?
D — A106205
E = ?

For example, the first would be used as follows:

f1(q) = q + 7 + 15 q + 71 q +

From alpha.ces.cwru.edu!somos Sun Aug 1 12:28:13 0400 1993
Subject: q-series calculation

Neil,

I finally broke down and automated the calculation. Here is the way I got the series and the results. In my opinion, these sequences are noteworthy and belong on your list. Note that aside from the last example using 163 which is based on the j function, I still have not identified the other sequences = series = functions. Shalom, Michael

---

```
(* Mathematica numerical calculation of q-series 1 Aug 1993 by Michael Somos *)

e[x_, n_] := Module[{y = Pi*Sqrt[x]}, N[Exp[-y], Ceiling[N[(n*y)/Log[10]]]]]
f[x_, q_, n_] := NestList[(#1 - Round[#1])/q & , x, n]
l[x_, s_, n_] := (q=e[Abs[x],n];TextForm[Round[f[s[q],Sign[x]q,n]]])

l[34/3,198*#^(1/2)&,12]
   {1, 7, 15, 71, 106, 273, 486, 961, 1563, 3040, 4692, 8199, 12774}
l[-59/3,1060*#^(1/2)&,21]
   {1, 5, 27, 41, 146, 243, 510, 887, 1755, 2728, 5052, 7857, 13157, 20253,
    32805, 48680, 76568, 112320, 169814, 246263, 365013, 519045}
l[-89/3,Sqrt[300*#^(1/3)]&,47]
   {1, 7, 8, 22, 42, 63, 106, 190, 267, 428, 652, 932, 1367, 2017, 2774,
    3950, 5539, 7541, 10342, 14184, 18889, 25435, 33974, 44720, 58952, 77550,
    100546, 130780, 169273, 217230, 278636, 356566, 452544, 574548, 726938,
    914742, 1149685, 1441787, 1798740, 2242436, 2788219, 3453787, 4272238,
l[58,396*#^(1/4)&,36]
   {1, 26, 79, 326, 755, 2106, 4460, 10284, 20165, 41640, 77352, 147902,
    263019, 475516, 816065, 1413142, 2353446, 3936754, 6391091, 10390150,
    16497734, 26184098, 40775677, 63394792, 97037170, 148178934, 223351867,
    335704742, 499050461, 739575640, 1085723797, 1588726100, 2305778480,
    3335492514, 4790460930, 6857634062, 9754445480}
l[-163,640320*#^(1/3)&,34]
```

A again A58206
F A58490
G A58537
H A52241

{1, 248, 4124, 34752, 213126, 1057504, 4530744, 17333248, 60655377, 197230000, 603096260, 1749556736, 4848776870, 12908659008, 33161242504, 82505707520, 199429765972, 469556091240, 1079330385764, 2426800117504, 5346409013164, 11558035326944, 24551042107480, 51301080086528, 105561758786885, 214100032685072, 428374478862400, 846173187465216, 1651298967150546, 3185652564830016, 6078963644150128, 11480231806541824, 21467177880529689, 39764843702689336, 72997137165153779}

I

You remember those q-series that I found empirically using the radix expansion of some numbers like exp(pi*sqrt(163))? Well I have finally bumped into print references to them. The July issue of MOC led me to them. In fact, there is a nice paper by Harvey Cohn that lists four coefficients for three variations of the j functions which includes the standard one. See page 158, table (3.2d).

The story doesn't end there. I seem to have recently uncovered another variation of j based on sqrt(5). Look at exp(pi*sqrt(38/5)) for the expansion. The coefficient list is {1,4,22,56,177,352,870,...} This was just yesterday. I still have to follow up the reference chain to see how much is known, but at least the Cohn article can be used as a print reference for three of the sequences. Shalom, Michael Somos

J

Subject: Monstrous Moonshine and q-series radix expansions
Status: RO

Dear Neil!

Greetings! I just today looked at "Monstrous Moonshine" for the first time (that I can remember) and what a revelation! Unlike most of the other published works in this area, this article actually had extensive tables with actual coefficients of the q-series. I did a quick survey on page 334 and found that all the series that I had just gotten a few weeks ago via radix expansions were listed! Since the reference to Fricke (which I haven't looked at yet) means that this kind of stuff has been published for over 100 years, it is just about time I knew about it. By the way, how many of those sequences of integers do you currently have in your Handbook? Shalom, Michael

Neil,

Sorry that you are sick. I had plenty to do while you took the time to reply, for which I am thankful. I have been entering data from table 4 of Monstrous Moonshine up to 10E so far. I realized that there was really not enough information available (extensive though it may appear at first glance) in the table.

For example, the actual modular function is not identified that well, unless I am overlooking the obvious. For example, what does 2B come from? In an attempt to investigate directly, I used all of the ten coefficients in the table and tried computing the function. In the process I came across the follwoing remarkable discovery.

Recall my original q-series article of earlier this month:

```
e[x_, n_] := Module[{y = Pi*Sqrt[x]}, N[Exp[-y], Ceiling[N[(n*y)/Log[10]]]]]
f[x_, q_, n_] := NestList[(#1 - Round[#1])/q & , x, n]
l[x_, s_, n_] := (q=e[Abs[x],n];TextForm[Round[f[s[q],Sign[x]q,n]]])
```

Now the following calculation is very remarkable:

```
l[7,Sqrt[2*#^(1/12)]&,95]
  {1, 1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 8, 8, 9,
   11, 12, 12, 14, 16, 17, 18, 20, 23, 25, 26, 29, 33, 35, 37, 41, 46, 49,
   52, 57, 63, 68, 72, 78, 87, 93, 98, 107, 117, 125, 133, 144, 157, 168,
   178, 192, 209, 223, 236, 255, 276, 294, 312, 335, 361, 385, 408, 437,
   471, 501, 530, 568, 609, 647, 686, 732, 784, 833, 881, 939, 1004, 1065,
   1126, 1199, 1279, 1355, 1433, 1523, 1621, 1717, 1814, 1926}
```

This looks fairly innocuous unless you realise just what it implies. By raising this series to the 24th power, you get essentially 2B in the table except for the alternation of signs:

A7191

2B

```
{1, 24, 276, 2048, 11202, 49152, 184024, 614400, 1881471,
   5373952, 14478180, 37122048, 91231550, 216072192, 495248952, 1102430208,
   2390434947, 5061476352, 10487167336, 21301241856, 42481784514,
   83300614144, 160791890304, 305854488576, 573872089212, 1063005978624,
   1945403602764, 3519965179904, 6300794030460, 11164248047616,
   19591528119288, 34065932304384, 58718797964805, 100372723007488,
   170215559855424, 286470013685760, 478625723149576, 794110053826560,
   1308745319975256, 2143055278039040, 3487563372381816, 5641848336678912,
   9074553043554568, 14515166263443456, 23093778743102262,
   36552977852071936, 57567784186189368, 90226777113575424,
   140752796480416011, 218578429975461888, 337945040343588276,
   520271697765971968, 797652526220573580, 1218002527825723392,
   1852604006634050072, 2807138079496716288, 4237760460302936433,
   6374456847628238848, 9554873766107770560, 14273181657059143680,
   21250450411204068910, 31535729115847852032, 46650835290143061624,
   68797209365301886976, 101150679669913197462, 148280443106626633728,
   216743142763626253712, 315923191441199824896, 459218611940943755226,
   665710603285072019456, 962508846974918603904, 1388038765923851599872,
   1996639069403279491427, 2864978197116521938944, 4100990608911708903432,
   5856297079648098807808, 8343432715970391209502, 11859696700297921757184,
   16820105145987654631552, 23802835313046730063872,
   33611779636250175278886, 47362494062244172660736,
   66600077798590855556532, 93460562353103053049856,
   130891485964083426534122, 182952844329494181838848,
   255227018229957765044016, 355376219286719031156736,
   493899311443420857952845, 685157678128482627354624,
   948763597225844233250504, 1311456320500974276980736,
   1809633323386495729057992, 2492760414984152205361152,
   3427959082742197097793024, 4706168520874397834575896}
```

Now, doesn't this give you a whole lot more information than what is in the table? I hope this information is of interest to you. Shalom, Michael
<somos@ces.cwru.edu>

To: njas@research.att.com
Subject: Ramanujan and Pi and modular equations
Status: R

Neil,

I was just looking at Ramanujan's collected papers, and his paper on Pi and modular equations has equations almost identical to my own, which he writes are similar to those of Weber. I can extend almost all of the sequences if enough non-zero terms are given to establish its\ unique identity and I can identify exact values (like integers) for the sum of the q series for particular values of q.

As for the Mathematica (note spelling) code, I could produce the equivalent code in Maple or Pari or anything else reasonable. What I essentially do is compute the value of q to enough accuracy to produce the number of terms of the sequence that I want. This is the purpose of the e[x_,n_] function. Then I iteratively pull "digits" in the radix expansion to radix q of some number that depends on q. This is the purpose of the f[x_,q_,n_] function. The final l[x_,s_,n_] function puts the two pieces together.

Look, you can write your own code to do the same thing. I can give the algorithm in standard mathematical notation as follows:

Given: r = a/b (some rational number). Let q = exp(-sqrt(r)) or perhaps the negative of that.
Given: x0 = c*|q|^(1/d) (for some positive integers c,d) such that x0 is approximately 1. Then define the sequence

s0 = round(x0) , x1 = (x0-s0)/q , s1 = round(x1) , x2 = (x1-s1)/q , ... where round(x) is the closest integer to x . The result is {s0, s1, s2, s3, ... }. In practice, stop when the terms of the sequence get bigger than 1/|q|. Just to give a very simple example of the general principle (inversion of a power series) look at the decimal expansion of 1/9899 . You should be able to recognize the pattern right away where the radix is 100.

Shalom, Michael Somos

From math.Princeton.EDU!conway Mon Aug 30 15:59:51 EDT 1993
Status: R

yes, they are pretty trivial to work out. How many of them you should give is a problem. Also, the q^0 coefficient is indeterminate in a certain sense. The value we give is what we call "the Rademacher value" - it is the only one that has any kind of "absolute" definition, but it is not always an integer, and in any case, no particular value is of real significance. So (for instance) the j-coefficients are

1, *, 196884, 21493760, ...

where "*" really is indeterminate. How should you index this? J

From alpha.ces.cwru.edu!somos Thu Sep 2 06:58:56 0400 1993
Received: by ninet.research.att.com; Thu Sep 2 07:00 EDT 1993
Received: by harry.CES.CWRU.Edu (5.64+/ane.09.11.90.2)
id AA05735; Thu, 2 Sep 93 06:58:56 -0400

Date: Thu, 2 Sep 93 06:58:56 -0400
From: Michael Somos <somos@alpha.ces.cwru.edu>
Message-Id: <9309021058.AA05735@harry.CES.CWRU.Edu>
To: njas@research.att.com
Subject: extending sequences
Status: R

Neil,

In a previous e-mail you asked "can you extend any of the other sequences in the same way?" The answer is yes, and I have already begun to do this, but it is a hard job. It would get especially hard for some of the later sequences with very small number of nonzero entries. Look at 24J for example, which is listed as all zeros after the initial 1. In any case, The tables 2 and 3 together are supposed to identify each modular function. I still have not deciphered the cryptic notation that they used. It seems easier for me (and more educational) to continue using my own methods. Here is what I have written so far about some of these sequences. Obviously there is a great amount which can be done and written about.

Modular Function Notes by Michael Somos
28 August 1993 CWRU

This is a list of empirical facts about certain special functions defined in the complex plane. The major source for identification of these kind of functions is Table 4 of "Monstrous Moonshine" by Conway and Norton which lists coefficients out to q^10 of head characters of elements of the Monster group. These just happen to be interesting functions on their own. The "Monstrous Moonshine" paper is on pages 308-339 of The Bulletin of the London Mathematical Society, October 1979 issue.

Let t be any complex number with positive imaginary part. The convention is q(t) = q = exp(pi*i*t) . Although j(t) is a function of q^2 , the intermediate expression need to use just q . The functions that will be considered have the general form:

```
j(t) = 1/q^2 + a0 + a1*q^2 + a2*q^4 + a3*q^6 + a4*q^8 + ...
```

Note: Table 4 lists values for a0 which are not natural in the context of this note and automorphic functions. I will use the value which makes more sense from the theory of functions of a complex variable.

Note: There are two common conventions for q. The other is exp(2*pi*i*t).

1A. The Klein absolute invariant

```
j(t) = 1/q^2 + 744 + 196884*q^2 + 21493760*q^4 + 864299970*q^6 + ...
```

This is the prototypical modular function. Volumes could be written about it alone. Here is quick summary of its invariance properties:

```
j(t) = j(-t) = j(t')' = j(t+1) = j(-1/t) ,  where (a+bi)' = a-bi .
```

A very efficient way of computing the value of this function is the result

```
j(t) = f(m(q(t)) , where  f(z) = 256*(z*z-z+1)^3/(z*z-z)^2 ,

m(z) = z*(a(z)/b(z))^4 ,  q(t) = exp(pi*i*t) , and

a(z) = 2 + 2*z^2 + 2*z^6 + 2*z^12 + ... + 2*z^(n*n+n) + ... ,
b(z) = 1 + 2*z^1 + 2*z^4 + 2*z^9  + ... + 2*z^(n*n) + ...  .
```

Geometrically, this function arises from functions on lattice shapes. A lattice in the complex plane is determined by two generators. The ratio of them is the lattice shape. Since lattices are not uniquely determined by two generators, we would like a function which produces the same value no matter which two generators are used. Explicitly, start with a lattice

```
L = { k1*w1 + k2*w2 : k1,k2 integers } ,  t = w1/w2 .
```

Now any other two generators can be given by using the linear transformation

```
(w1,w2) --> (a*w1+b*w2,c*w1+d*w2) ,  a*d-b*c = 1 ,  a,b,c,d integers .
```

This gives a linear fractional transformation of the lattice shape t, thus

```
t --> (a*t+b)/(c*t+d) .
```

The group of all such transformations is known as the modular group and denoted SL_2(Z) . It is generated by the two transformations

```
t --> t+1 , and  t --> -1/t .
```

This is the reason for the last two invariance properties of j . The other two are simpler. The first is based on the fact that negating a lattice generator yields the same lattice, and the second is a general fact about complex conjugation. As a consequence of these invariance properties, the fundamental region can be chosen as a strip of width one centered at the origin and exterior to the unit circle. The imaginary axis from sqrt(-1) to infinity is mapped to the reals from 1728 upward. The unit circle from sqrt(-1) to (1+sqrt(-3))/2 is mapped to the interval from 1728 to 0 . And the ray above (1+sqrt(-3))/2 is mapped to the negative reals.

Some exact values for j(t):

t	j(t)	
sqrt(-1)	1728	= 12^3
sqrt(-2)	8000	= 20^3
sqrt(-3)	54000	= 30^3 * 2
sqrt(-4)	287496	= 66^3
sqrt(-7)	16581375	= 255^3
(1+sqrt(-3))/2	0	
(1+sqrt(-7))/2	-3375	= -15^3
(1+sqrt(-11))/2	-32768	= -32^3
(1+sqrt(-19))/2	-884736	= -96^3
(1+sqrt(-27))/2	-12288000	= -160^3 * 3
(1+sqrt(-43))/2	-884736000	= -960^3
(1+sqrt(-67))/2	-147197952000	= -5280^3
(1+sqrt(-163))/2	-262537412640768000	= -640320^3

2A. The invariant of G(sqrt(2)) ~~A7267~~

2A

j2A(t) = 1/q^2 + 104 + 4372*q^2 + 96256*q^4 + 1240002*q^6 + ...

do it

Some exact values for j2A(t):

t	j2A(t)	
sqrt(-2)/2	256	= 4^4
sqrt(-4)/2	648	= 3^4 * 8
sqrt(-6)/2	2304	= 4^4 * 9
sqrt(-10)/2	20736	= 12^4
sqrt(-18)/2	614656	= 28^4
sqrt(-22)/2	2509056	= 12^4 * 121
sqrt(-58)/2	24591257856	= 396^4
(1+sqrt(-1))/2	0	
(1+sqrt(-3))/2	-144	= - 12^2
(1+sqrt(-5))/2	-1024	= - 32^2
(1+sqrt(-7))/2	-3969	= - 63^2
(1+sqrt(-9))/2	-12288	= - 64^2 * 3
(1+sqrt(-13))/2	-82944	= - 288^2
(1+sqrt(-25))/2	-6635520	= - 1152^2 * 5
(1+sqrt(-37))/2	-199148544	= - 14112^2