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ON THE NUMBER OF PLANAR MAPS

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1. Introduction. In a survey of methods in enumerative map theory [**14**], W. T. Tutte pointed out that little has been done towards enumerating unrooted maps other than plane trees. A notable exception is to be found in the work of Brown, who took the initial step in this direction by enumerating non-separable maps up to sense-preserving homeomorphisms of the plane [**2**]. He then took a further step, allowing sense-reversing homeomorphisms, by counting triangulations and quadrangulations of the disc [**3**, **4**]. In all these problems, however, there is a fixed outer region of the plane. This can be considered as a certain type of rooting of a planar map, which is normally regarded as lying on the sphere or closed plane. It is our object here to find an expression for the number of unrooted planar maps in a given set, in terms of the numbers of maps in that set which have been rooted in a special way.

The principal methods used in enumerating unlabelled graphs involve a suitable application of Burnside's Lemma, either explicitly or implicitly in the guise of the elegant counting theorem developed by Pólya [**11**]. Abundant examples of this technique are to be found in [**9**], along with similar techniques based on extensions or implications of Burnside's Lemma. However, if Burnside's Lemma is to be used to enumerate a set of configurations, the configurations must fall into equivalence classes defined by the orbits of some group of permutations of the "labelled" configurations. The point of the labelling is to destroy all non-trivial automorphisms of a configuration. In the graphical case, the p points of a graph are labelled with the numbers from 1 to p to obtain a labelled graph. The group which is used corresponds to the group of permutations of the labels. In the case of maps, the natural choice for the "labelling" is the rooting of a map; that is, the specification of an edge, together with a direction along that edge and a direction across the edge. However, it is difficult to find a suitable group of permutations with which to work. We have overcome this obstacle, but in the process the part played by Burnside's Lemma has become so insignificant that an elementary argument seems more appropriate. Nevertheless, the main theorem in Section 3 retains a strong resemblance to Burnside's Lemma. The result is manipulated in Section 4 into a form which is more complicated but more useful in specific applications. The enumeration of maps possessing a sense-reversing symmetry is discussed in Section 5.

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The quest for a formula giving, or at least an algorithm for calculating, the number of combinatorially distinct convex polyhedra with p vertices has a long history, which apparently began with Euler and was recently surveyed in [**6**]. On the other hand, the problem of finding the number of planar graphs on p points was first posed relatively recently in [**7**], and has remained a tantalizing unsolved problem in its own right. Both these problems can be solved by applying the formulae of the present paper, but the mechanism of such applications, as well as the application to enumerating all planar maps, has little to do with our present objective and will be treated elsewhere.

2. Preliminaries. The basis for a rigorous combinatorial theory of maps was established by Tutte [**13**], but this approach is rather unwieldy for our present purposes. We prefer to retain a mixture of combinatorial and topological concepts.

The graph theoretic notation of [**8**] is assumed, whilst for fundamental planarity concepts we follow the approach of [**12**]. In particular, a *planar map* M is the dissestion of a 2-sphere (or, alternatively, a closed plane) Π determined by a proper drawing of a connected pseudograph G. By this it is meant that the points and lines of G are disjoint subsets of Π, called respectively the *vertices* and *edges* of M, such that a line x of G is an open arc in Π whose endpoints are the points of G incident with x. We refer to M simply as a *map*, and to G as the *pseudograph* of M. The *faces* of M are the components of the subset of Π formed on deletion of all vertices and edges of M, each face being homeomorphic to an open dis

The vertices, faces and edges of M are its *cells*, two of which are *incident* if one is contained in the boundary of the other. A *loop* of M is an edge corresponding to a loop of G; that is, an edge whose endpoints coincide. A *bridge* of M is an edge corresponding to a bridge of G (a line whose removal disconnects G). The *valency* of a vertex v is the number of edges with which v is incident, where a loop adjacent to v is counted twice. Similarly, the valency of a face f is the number of edges with which f is incident, where a bridge incident with f is counted twice. The valency of an edge is 2.

A *homeomorphism* of a map M_1 onto a map M_2 is a homeomorphism of Π onto itself which maps the vertices, edges and faces of M_1 onto the vertices, edges and faces respectively of M_2. If two maps are homeomorphic to one another, we shall henceforth consider them to be the same map.

For any map M, another map M^* can be constructed subject to the following conditions:

(i) the vertices of M^* are contained in the faces of M, one in each face;
(ii) each edge of M has nonempty intersection with just one edge of M^*;
(iii) the above two conditions with M and M^* interchanged.

Such a map M^* is called the *dual* of M: the facts that M has a unique dual M^*, and that the dual of M^* is M, are readily proved using the Jordan Curve Theorem together with elementary topological arguments. In this paper we shall often omit such proofs.

An *end* of an edge x of a map can be regarded as a connected portion of x in a (sufficiently small) neighbourhood of one of the incident vertices of x. Thus each edge, even a loop, has two distinct ends. An edge x is *directed* by specifying one of its ends. It simplifies our discussion if we assume that Π has been oriented. This assumption has no effect on our enumerative results, because two maps are considered the same whenever there is a sense-reversing homeomorphism from one to the other. With respect to the given orientation of Π we may refer to the right and left sides of a directed edge. An *arrow* of a map M is an ordered triplet $A = \{x, x', x''\}$ where x is an edge of M, x' is an end of x and x'' is a side of x. Let f be the face of M which is incident with x and lies on the x'' side of x, and let v be the vertex which lies at the x' end of x. Then A is said to be incident with x, with f and with v.

A map with e edges has $4e$ arrows. A *rooted* map is a map in which one of these arrows, called *root arrow*, has been distinguished from the rest; that is, there is a specified edge with a specified side and a specified end. The vertex, edge and face incident with the root arrow are called the *root vertex*, *root edge* and *root face* respectively.

Two homeomorphisms h_1 and h_2 of a map M onto itself are considered equivalent if $h_1 h_2^{-1}$ maps each arrow of M onto itself. The equivalence classes of this relation are called *automorphisms* of M. Each automorphism α of M is thus a set of homeomorphisms of Π, and α can be represented as an isomorphism of the set of arrows of M onto itself. An automorphism of M induces an isomorphism of each of the following sets onto itself: the sides of the edges of M; the ends of the edges of M; the edges of M; the vertices of M; the faces of M.

Two arrows A_1 and A_2 of a map M are *similar* if there is an automorphism of M which maps A_1 onto A_2. A map becomes *twice-rooted* if an ordered pair $\{A_1, A_2\}$ of similar (but not necessarily distinct) arrows are distinguished. The arrows A_1 and A_2 are called the *primary* and *secondary* root arrows respectively. The *underlying rooted map* of a twice-rooted map is the rooted map obtained by distinguishing the primary root arrow alone. The *underlying map* of a twice-rooted map, or a rooted map for that matter, is just the unrooted version of the map.

For maps with distinguished arrows, the additional requirement is adjoined to the definition of a homeomorphism, that it must preserve the set of distinguished arrows. In the case of twice-rooted maps, the ordering of the distinguished arrows must also be preserved.

A rooted map can be represented in a drawing by placing a diagrammatical arrow on the specified side of the root edge, pointing towards the specified end. For example, the four ways to root one of the maps with one edge are shown in Figure 1. All four rootings produce the same rooted map.

FIGURE 1. The four rootings of a map.

The following result is well-known and was apparently first used in full generality in [**12**].

LEMMA 1 (Tutte). *The only automorphism of a rooted map is the identity.*

This has an immediate consequence for twice-rooted maps.

LEMMA 2. *The only automorphism of a twice-rooted map is the identity.*

3. The main theorem. If $\mathscr{C}$ is a set of maps, $\mathscr{T}(\mathscr{C})$ is defined to be the set of all twice-rooted maps whose underlying maps are elements of $\mathscr{C}$. We denote the cardinality of $\mathscr{C}$ by $|\mathscr{C}|$.

THEOREM 1. *Let $\mathscr{C}$ be a set of distinct maps having $e \geqq 1$ edges each. Then*

$$|\mathscr{C}| = \frac{1}{4e} |\mathscr{T}(\mathscr{C})|.$$

Proof. Let M be a map in $\mathscr{C}$, and let $\mathscr{R}$ be the family (with multiplicities retained) of rooted maps obtained by distinguishing the arrows of M one by one. If R is a rooted version of M, the number $n(R)$ of times that R appears in $\mathscr{R}$ is just the number of arrows of R which are similar to its distinguished arrow. By Lemma 2, this in turn is equal to the number of twice-rooted maps whose underlying rooted map is R. Summing over all rooted versions R of M, we obtain

$$4e = |\mathscr{R}| = \sum_{R \in \mathscr{R}} n(R) = |\mathscr{T}(\{M\})|,$$

so that

$$1 = \frac{1}{4e} |\mathscr{T}(\{M\})|.$$

The theorem follows upon summation of this equation over all M in $\mathscr{C}$.

4. Special maps. An arrow $A = \{x, x', x''\}$ is said to have *positive sense* if x'' is the right side of the directed edge obtained by specifying the x' end of x, and *negative sense* otherwise. A sense-preserving homeomorphism of a map M will map each arrow onto one of the same sense, whilst a sense-reversing homeomorphism will negate the sense of every arrow. We may speak of an automorphism as being sense-preserving or sense-reversing, according to the same rule.

A cell c of a map M is an *invariant* of an automorphism α of M if $\alpha(c) = c$. The following result has been used from time to time, in one form or another, for various classes of maps; for examples, see [**1**] and [**15**].

LEMMA 3. *If α is a sense-preserving automorphism of a map M, then either α has just two invariants or α is the identity.*

Proof. It is shown in [**10**] and [**5**] that every non-identity sense-preserving transformation of the sphere with finite period is topologically equivalent to a rotation. In particular, this implies that any such transformation f has precisely two fixed points. Given an automorphism α of M, it is not difficult to define a homeomorphism f in the equivalence class α such that f has just one fixed point in a cell c whenever c is an invariant of α. The lemma follows.

The equation given in Theorem 1 can be manipulated, via Lemma 3, into a form which is much harder to express but easier to apply. We first introduce the necessary notation.

A *fully-rooted* map $(M, \mathscr{A})$ consists of an underlying map M together with a distinguished set $\mathscr{A}$ of arrows of M, such that $\mathscr{A}$ is an orbit of some automorphism of M. This fully-rooted map is *sensed* if all arrows in $\mathscr{A}$ have the same sense, and *unsensed* otherwise. A *sensed special* map $(M, \mathscr{A}, c)$ is a sensed fully-rooted map $(M, \mathscr{A})$ with a distinguished cell c, called the *root* of the special map, such that each arrow in $\mathscr{A}$ is incident with the root. If $\mathscr{A}$ contains just one arrow, the restriction is imposed that the root must be the incident edge. An *unsensed special* map $(M, \mathscr{A}, c_1, c_2)$ is an unsensed fully-rooted map $(M, \mathscr{A})$ with two (unordered) distinguished cells c_1 and c_2 of the same dimension, called the *roots* of the special map, such that each arrow in $\mathscr{A}$ is incident with c_1 or c_2 or both. If $\mathscr{A}$ contains just two arrows, the further restriction is imposed that the roots must be the incident edges. The roots may coincide.

A sensed special map is classified as *type* 1 if it has just one distinguished arrow, in which case the special map can be regarded as an ordinary rooted map. A sensed special map $F = (M, \mathscr{A}, c)$ in which $\mathscr{A}$ contains more than one arrow is classified as *type* 2 (*type* 3 or *type* 4) if c is a face (edge or vertex, respectively) of M.

Note that a sensed fully-rooted map will correspond to more than one special map whenever more than one cell is eligible to be specified as the root. Of course, the resulting special maps must be of different "types". One such fully-rooted map is exhibited in Figure 2 (iii).

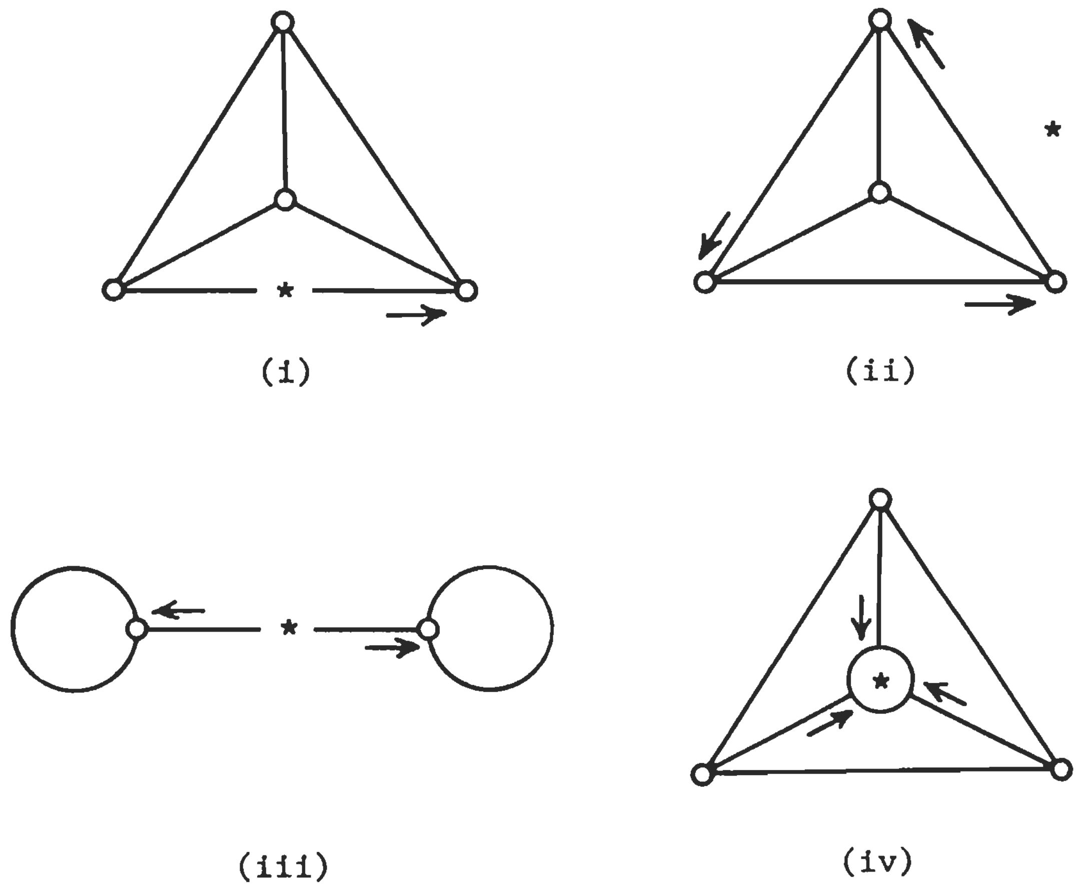

FIGURE 2. Four types of sensed special maps. The root is denoted "*".

The number of distinguished arrows in an unsensed special map $F = (M, \mathscr{A}, c_1, c_2)$ is a multiple of 2. We classify F as *type* 1 if $\mathscr{A}$ contains just two arrows, and otherwise F is *type* 2 (*type* 3 or *type* 4) if the roots c_1 and c_2 are faces (edges or vertices, respectively) of M.

An (s, k)-*special* map is a sensed or unsensed special map of type 2, 3 or 4 in which the roots have valency s, and precisely k distinguished arrows are incident with each root.

Let $\mathscr{C}$ be a set of maps, and let $\mathscr{F}$ denote the set of special maps whose underlying maps are elements of $\mathscr{C}$. We denote by $a_i(\mathscr{C})$ the number of type i sensed special maps in $\mathscr{F}$, and $b_i(\mathscr{C})$ is defined the same way for unsensed special maps. For $i \neq 1$, the number of type i sensed (or unsensed) (s, k)-special maps in $\mathscr{F}$ is denoted $a_i(\mathscr{C}; s, k)$ (or $b_i(\mathscr{C}; s, k)$ respectively). Finally, let

(1) $a(\mathscr{C}; s, k) = \sum_{i=2}^{4} a_i(\mathscr{C}; s, k)$

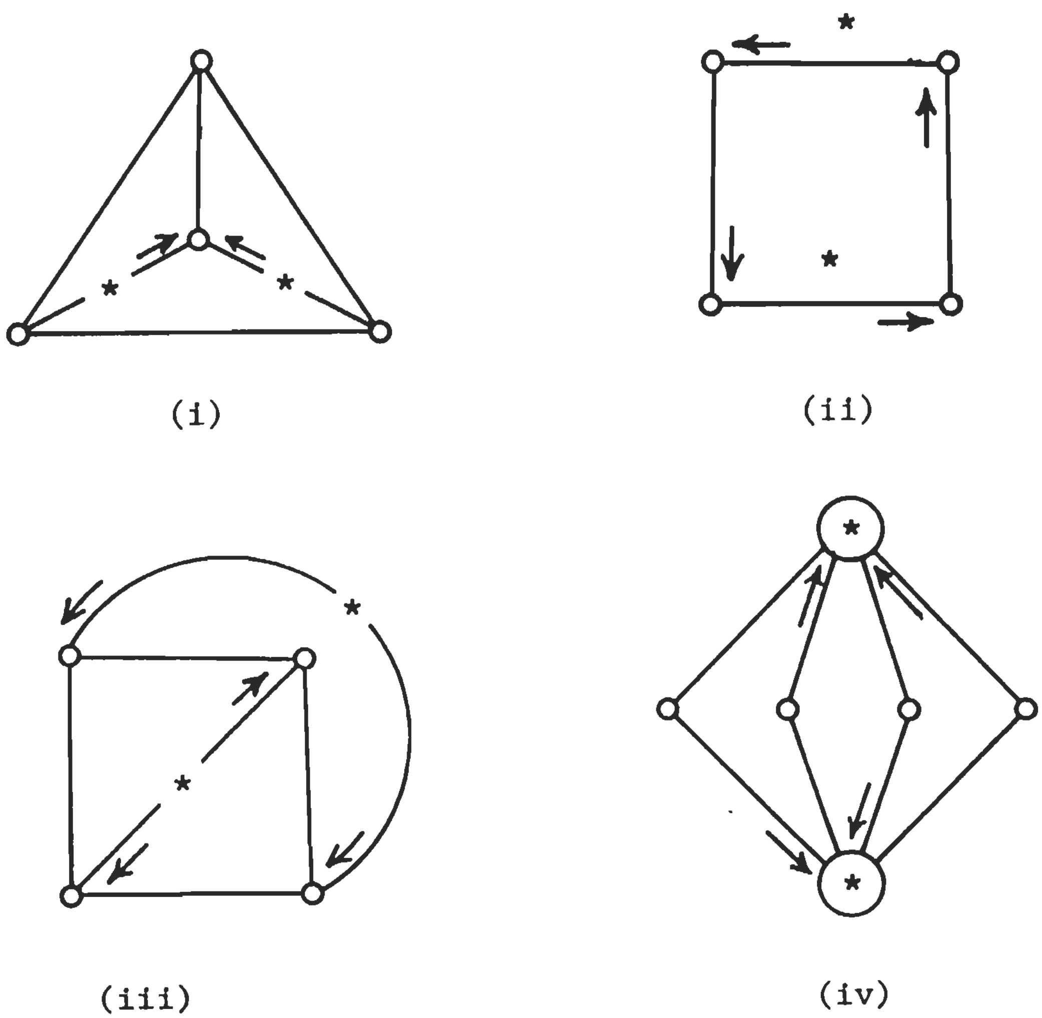

FIGURE 3. The four types of unsensed special maps.

and

$$(2) \quad b(\mathscr{C}; s, k) = \sum_{i=2}^{4} b_i(\mathscr{C}; s, k).$$

THEOREM 2. *Let $\mathscr{C}$ be a set of distinct maps having $e \geqq 1$ edges each. Then*

$$|\mathscr{C}| = \frac{1}{4e}[a_1(\mathscr{C}) + b_1(\mathscr{C})] + \sum_{s=2}^{2e} \frac{1}{4s} \sum_{\substack{k \mid s \\ k \geqq 2}} [\phi(k) a(\mathscr{C}; s, k) + \phi(2k) b(\mathscr{C}; s, k)]$$

where ϕ denotes the Euler function.

Proof. Given a map M in $\mathscr{C}$ and a twice-rooted map T in $\mathscr{T}(\{M\})$, by Lemma 1 there is a unique automorphism α_T of M which maps the primary root arrow of T onto its secondary root arrow. We partition $\mathscr{T}(\{M\})$ into four parts $\mathscr{T}_i$ by placing T in

$\mathscr{T}_1$ if α_T is the identity,
$\mathscr{T}_2$ if α_T is sense-preserving but not the identity,
$\mathscr{T}_3$ if α_T is sense-reversing and α_T^2 is the identity,
$\mathscr{T}_4$ otherwise.

We have by Theorem 1 that

$$(3) \quad 4e = \sum_{i=1}^{4} |\mathscr{T}_i|.$$

Suppose we choose an element T of $\mathscr{T}_2$. Then by Lemma 3, α_T has just two invariants. We choose one of these invariants, say c. Let S be the set of arrows of M incident with c. Then $|S| = 2s$, where s is the valency of c. The set S is a union of orbits of α_T. These orbits contain k arrows each, where k is the order of α_T. Denote the orbits by $\mathscr{A}_1, \ldots, \mathscr{A}_{2s/k}$. We now draw a picture of each of the sensed (s, k)-special maps $(M, \mathscr{A}_1, c), \ldots, (M, \mathscr{A}_{2s/k}, c)$ and assign a weight of $\frac{1}{2}k/s$ to each of the drawings so obtained. The whole process is performed once for each of the $2|\mathscr{T}_2|$ choices for the twice-rooted map T in $\mathscr{T}_2$ and the invariant c of α_T. The sum, w say, of the weights of the drawings we have made is given by

$$(4) \quad w = 2|\mathscr{T}_2|.$$

Given a sensed (s, k)-special map $F = (M, \mathscr{A}, c)$, there are just $\phi(k)$ automorphisms β of F such that $\mathscr{A}$ is an orbit of β. As each such automorphism leaves c fixed and has $4e/k$ orbits, the number of times we have drawn F is just $4\phi(k)e/k$. The number of arrows in S with a given sense is just s, so that $k|s$ and $s \geqq 2$. Hence, on summing the weights of the drawings for each s and k, we obtain

$$(5) \quad w = \sum_{s=2}^{2e} \frac{2e}{s} \sum_{\substack{k|s \\ k \geqq 2}} \phi(k) a(\{M\}; s, k).$$

Suppose we choose an element T of $\mathscr{T}_4$. Then by Lemma 3, α_T^2 has just two invariants, say c_1 and c_2. Since α_T^2 is not the identity and α_T is sense-reversing, neither c_1 nor c_2 is an invariant of α_T. Since $\alpha_T(c_i)$ is an invariant of α_T^2 for each i, it follows that α_T interchanges c_1 and c_2. In particular, c_1 and c_2 have the same dimension, so that no arrow of M can be incident with both c_1 and c_2. Let S be the set of arrows incident with either c_1 or c_2. Then S contains $4s$ arrows, where s is the valency of c_1 and of c_2. The orbits of α_T contain $2k$ arrows each, where k is the order of α_T^2. Denote the orbits of α_T contained in S by $\mathscr{A}_1, \ldots, \mathscr{A}_{2s/k}$. As before, we make a drawing of each of the (s, k)-special maps $(M, \mathscr{A}_i, c_1, c_2)$ and assign a weight of $\frac{1}{2}k/s$ to each of the drawings so obtained. This time, the process is performed just once for each element T of $\mathscr{T}_4$. It follows that the sum of the weights of the drawings is $|\mathscr{T}_4|$.

Let $F = (M, \mathscr{A}, c_1, c_2)$ be any sensed (s, k)-special map whose underlying map is M. Then $\mathscr{A}$ is an orbit of some sense-reversing automorphism α of M. As the roots c_1 and c_2 are each incident with an arrow in $\mathscr{A}$, neither root can be an invariant of α; otherwise α^2 would be the identity. Hence $c_1 \neq c_2$ and α interchanges c_1 and c_2. It follows that the number of times we have drawn F is $2\phi(2k)e/k$, so we have

$$(6) \quad |\mathscr{T}_4| = \sum_{s=2}^{2e} \frac{e}{s} \sum_{\substack{k \mid s \\ k \geqq 2}} \phi(2k) b(\{M\}; s, k).$$

As $|\mathscr{T}_1| = a_1(\{M\})$ and $|\mathscr{T}_3| = b_1(\{M\})$, equations (3) through (6) imply

$$4e = a_1(\{M\}) + b_1(\{M\}) + \sum_{s=2}^{2e} \frac{e}{s} \sum_{\substack{k \mid s \\ k \geqq 2}} [\phi(k) a(\{M\}; s, k) + \phi(2k) b(\{M\}; s, k)].$$

On dividing by $4e$ and summing over all M in $\mathscr{C}$, we obtain the theorem.

We say $\mathscr{C}$ is *closed under duality* if the dual of each map in $\mathscr{C}$ is also in $\mathscr{C}$. For such a set of maps, it is not necessary to consider the type 4 special maps.

THEOREM 3. *Let $\mathscr{C}$ be a set of distinct maps with $e \geqq 1$ edges each, which is closed under duality. Then*

$$|\mathscr{C}| = \frac{1}{4e} [a_1(\mathscr{C}) + b_1(\mathscr{C})] + \frac{1}{8} a_3(\mathscr{C}) + \frac{1}{4} b_3(\mathscr{C}) + \sum_{s=2}^{2e} \frac{1}{2s} \sum_{\substack{k \mid s \\ k \geqq 2}} [\phi(k) a_2(\mathscr{C}; s, k) + \phi(2k) b_2(\mathscr{C}; s, k)].$$

Proof. Duality provides a one-to-one correspondence between the type 2 sensed (s, k)-special maps whose underlying maps are in $\mathscr{C}$, and those of type 4. The same holds for the unsensed varieties. Hence $a_2(\mathscr{C}; s, k) = a_4(\mathscr{C}; s, k)$ and $b_2(\mathscr{C}; s, k) = b_4(\mathscr{C}; s, k)$. We also observe that

$$a_3(\mathscr{C}; s, k) = \begin{cases} a_3(\mathscr{C}) & \text{if } s = k = 2 \\ 0 & \text{otherwise} \end{cases}$$

and a corresponding observation can be made of $b_3(\mathscr{C}; s, k)$. The theorem now follows from Theorem 2.

Suppose we wish to enumerate a set $\mathscr{C}$ of unrooted maps. Theorems 2 and 3 express the number of maps in $\mathscr{C}$ which have e edges in terms of the numbers of various classes of special maps with e edges. The advantage of these results over Theorem 1 is due to the fact that if all the edges incident with the distinguished arrows of a special map F are merged with their incident faces, the resulting object retains the automorphisms of F. For certain sets $\mathscr{C}$, this enables the calculation of the numbers of special maps in terms of the numbers of special maps with smaller numbers of edges, thereby producing a recurrence relation for these numbers.

5. Achiral maps. A map is *chiral* if all its automorphisms are sense-preserving, and *achiral* otherwise. A *sensed* map consists of an underlying map M in which the set of positive arrows of M is specified. (Alternatively, it is possible to arrive at the notion of a sensed map by regarding two planar maps as the same only if there is a sense-preserving homeomorphism from one to the other.) This definition does not clash with the definition of a sensed special map $F = (M, \mathscr{A}, c)$ if we specify that the arrows in $\mathscr{A}$, and all others with the same sense, are positive.

If $\mathscr{C}$ is a set of maps, let $r(\mathscr{C})$ be the number of sensed maps whose underlying maps are elements of $\mathscr{C}$, and $n(\mathscr{C})$ the number of achiral maps in $\mathscr{C}$. The following result allows one to find $n(\mathscr{C})$, given $|\mathscr{C}|$ and $r(\mathscr{C})$.

LEMMA 4. *The number of maps in* $\mathscr{C}$ *is* $\frac{1}{2}(r(\mathscr{C}) + n(\mathscr{C}))$.

Proof. Every chiral map is the underlying map of two sensed maps, and every achiral map is the underlying map of just one sensed map. Hence if $d(\mathscr{C})$ is the number of chiral maps in $\mathscr{C}$, we have

$$r(\mathscr{C}) = n(\mathscr{C}) + 2d(\mathscr{C})$$

and

$$|\mathscr{C}| = n(\mathscr{C}) + d(\mathscr{C}).$$

We next examine the calculation of $r(\mathscr{C})$ for a given set $\mathscr{C}$. Minor modifications of the arguments in Sections 3 and 4 allow us to make the following assertion of a set $\mathscr{C}$ of distinct maps having $e \geqq 1$ edges each.

THEOREM 4. *The number of sensed maps whose underlying maps are in* $\mathscr{C}$ *is given by*

$$r(\mathscr{C}) = \frac{1}{2e} a_1(\mathscr{C}) + \sum_{s=2}^{2e} \frac{1}{2s} \sum_{\substack{k \mid s \\ k \geqq 2}} \phi(k) a(\mathscr{C}; s, k).$$

Just as $r(\mathscr{C})$ is obtained in terms of the a's, when Lemma 4 and Theorems 2 and 4 are combined we obtain $n(\mathscr{C})$ in terms of the b's.

THEOREM 5. *The number of achiral maps in* $\mathscr{C}$ *is given by*

$$n(\mathscr{C}) = \frac{1}{2e} b_1(\mathscr{C}) + \sum_{s=2}^{2e} \frac{1}{2s} \sum_{\substack{k \mid s \\ k \geqq 2}} \phi(2k) b(\mathscr{C}; s, k).$$

Of course, results corresponding to Theorem 3 hold for both $r(\mathscr{C})$ and $n(\mathscr{C})$.

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