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if you wish to exclude them from the answer, then we are left with the two following rules:

> If N is even, then let $N = 2q$. The number of triangles with perimeter N is T_{q-1}. For example, if $N = 10$, then there are $T_4 = 10$ such triangles.
>
> If N is odd, then let $N = 2q + 1$. The number of triangles with perimeter N is T_q. For example, if $N = 7$, then there are $T_3 = 6$ such triangles.

In each case our rules will count all triangles, including rotations and reflections.

A MODERN COMBINATORIAL PROBLEM

When first looking at Alcuin's puzzle of sharing barrels, we came across a curious concept. "The more interesting question," we said, "is not to find any one way of partitioning the barrels, but to find the total *number* of ways of doing so." This is the first hint of a whole area of dizzying puzzles called combinatorial puzzles.

An example of a fairly difficult combinatorial puzzle is this: In how many ways can q objects be partitioned into at most three parts? Let $q = 6$; then we have the following partitions:

(1) 6		
(2) 1 + 5	(3) 2 + 4	(4) 3 + 3
(5) 1 + 1 + 4	(6) 1 + 2 + 3	(7) 2 + 2 + 2

In other words, there are 7 ways to partition six objects into at most three parts. Finding these partitions is itself a little bit of a trick, requiring a fairly clear head. Notice that we do not include such partitions as 2 + 3 + 1, since this is the same as 1 + 2 +3, and that we *do* include 1 + 5 as a partition, since we want *at most* three parts.

Here is a brief table for readers who wish to check their answers while partitioning other sets of q objects into at most three parts. The fun job of actually finding the partitionings I leave to you:

q	Number of Partitions
0	1
1	1
2	2
3	3
4	4
5	5
6	7
7	8
8	10
9	12
10	14
11	16
12	19
13	21

The series in the table on the left we will call N_q. For example, $N_0 = 1$, $N_1 = 1$, $N_2 = 2$, $N_3 = 3$, and so on. Deriving a general formula for this series is enormously difficult. The formula is recursive, meaning that each new number depends on a value of a previous number. More precisely, $N_{q+6} = N_q + q + 6$. You can verify a few examples. Looking at the end of the table, you can see that $N_{13} = 21$, so a set of 13 objects should be partitioned in 21 different ways.

It turns out that this series becomes important in answering yet another aspect of Alcuin's barrel-sharing puzzle, and its cousin, the triangle puzzle. Remember that earlier we gave only the total number of solutions to the puzzle, including permutations and "degenerate" answers. Remember also that the unique, nondegenerate solutions all tucked themselves into a corner of the upside-down triangle. It is reasonable to ask for a count of these solutions. This we do in the table on the right, for each value of the number of barrels.

This series we will call A_q, in honor of Alcuin. For example, $A_5 = 1$, $A_6 = 2$, $A_7 = 1$, and so on.

Number of Barrels	Number of Unique non-Degenerate Solutions
5	1
6	2
7	1
8	3
9	2
10	4
11	3
12	5
13	7
14	7
15	5
16	8
17	7

The series on the right we will call A_q, in honor of Alcuin. That is, $A_5 = 1, A_6 = 2, A_7 = 1$, and so on. Is there any order to this series at all? Yes, and the order may be found in the previous series N_q. Look at the odd positions in the series, that is, $A_5, A_7, A_9. \ldots$ We have 1, 1, 2, 3, 4, 5, 7 . . . This is simply the series N_q starting at $q = 0$.

Now look at the even positions in the series, that is, $A_6, A_8, A_{10}. \ldots$ We have 2, 3, 4, 5, 7 . . . This again is the series N_q, this time starting at $q = 2$.

Thus, the series A_q is actually an interleaving of the series N_q. And Alcuin's barrel-sharing puzzle is an interleaving of a more serious combinatorial problem. It is almost mind-boggling to see how so many seemingly unrelated problems come together. Here again, credit for bringing everything together belongs to David Singmaster.

POURING WINE ON A RHOMBOID POOL TABLE

We have seen that Alcuin's problem of sharing barrels was related, at some level, to two other problems, although sometimes the relationship was well hidden. It is always a great pleasure to find this, since it hints that perhaps deep down, at some sublime level of abstraction, all puzzles are essentially the same. Here another classic puzzle comes to mind, although this time the kinship is not in the spirit of the puzzle, but in the method of solution. It was invented by a sixteenth-century Italian mathematician, Niccolò Fontana, known by his nickname, Tartaglia, "the Stutterer." (Tartaglia claimed that as a young boy, when Italy's wealth was being sacked by invaders, he developed his severe speech defect when a French soldier slashed his face. All writers take this story seriously, but of course it is nonsense to think a single incident, no matter how frightening, could effect a lifelong stammer.) Here is Tartaglia's problem:

> Three containers measure 3, 5, and 8 quarts respectively. The first two are empty, but the last is filled with wine. By pouring the wine from one container to another without ever losing any, and using no other measures, is it possible to end up with exactly two equal measures of wine?

(We will soon see a very simple graphic solution of this problem, so

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$A_{2n-1} = g_{n-3}$ $n \geq 2$

$A_{2n} = g_{n-1}$ $n \geq 2$

n	g_n	A_n
0	1	
1	1	
2	2	
3	3	
4	4	
5	5	1
6	7	2
7	8	1
8	10	3
9	12	2
10	14	4
11	16	3
12	19	5
13	21	4
14	24	7
15	27	5
16	30	8
17	33	7
18	37	10
19	40	8
20	44	12
21	48	10
22	52	14
23	56	12
24	61	16
25	65	14
26	70	19
27	75	16

1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, 12, 10,

14, 12, 16, 14, 19, 16, 21, 19, 24, 21, 27, 24, 30, 27,

33, 30, 37, 33, 40, 37, 44, 40, 48, 44, 52, 48, 56,

52, 61, 56, 65, 61, 70, 65, 75, 70, 80, 75, 85, 80, 91,

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