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Watson

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# ON "LEGO" TOWERS

P. J. S. WATSON
*Zaria, Nigeria*

We present four problems relating to the building toy "Lego Blocks." Two are solved analytically, and we present numerical solutions to the other two.

The building toy, "Lego Blocks," must be known to everyone: if it is unfamiliar to the reader, we can only suggest he obtain a small set. We define a tower as being constructed from the smaller common size of blocks, 2 units by 2 units, with only one block on each level, connected in the obvious sense. A stable tower is one which is stable against small displacements and each block is connected to its neighbors by at least two lugs. There are $3^{n-1}$ possible order $n$ towers. The five pictured below, in side view, together with mirror images of the first four, comprise the order 3 towers.

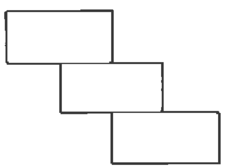

Figure 1

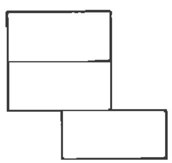

Figure 2

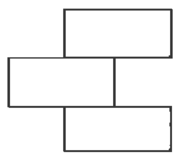

Figure 3

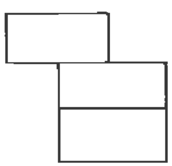

Figure 4

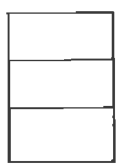

Figure 5



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Of these only Figure 1 is not stable. We can say that Figure 4 (inter alia) has an overhang of one unit and Figure 5 is the most stable tower, with its center of mass immediately above the center of the lowest block, so it must be tilted through the largest angle before falling.

One must be rather careful about the definition of "small displacement": suppose one has a tower of the form in Figure 6 with $n$ blocks in the upper part. The center of gravity will be at a point with coordinates

$$\left(1 - \frac{1}{n}, \frac{n+1}{2}\right).$$

(Though not quite true with Lego, we assume the blocks are twice as wide as they are high.) Hence the tower will fall if tilted through an angle $\theta$

$$\theta > \theta_n = \tan^{-1}\frac{2}{n(n+1)}$$

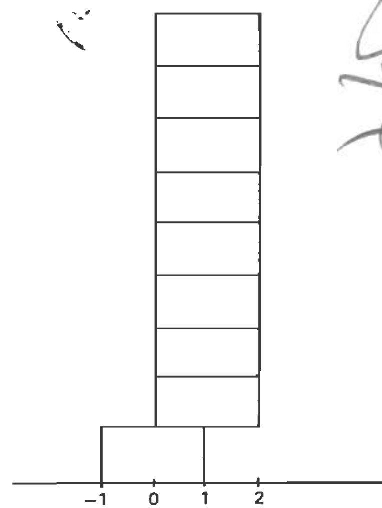



Figure 6

Obviously as $n$ becomes large, this angle becomes very small. However we must still define this tower as stable, as it can be tilted through an angle $\theta < \theta_n$ and will remain standing. Use of real (as opposed to mathematical) Lego Blocks limits this kind of tower to about eight blocks in height.

The four problems are:

(1) How high must a stable tower be to overhang $n$ units, if we insist that it lean monotonically to one side?

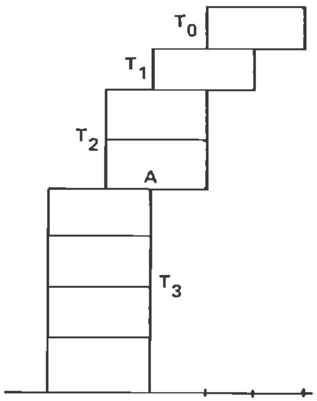
(2) How high must a stable tower be to overhang $n$ units, if we relax this restriction?

(3) What number, $S_n$ of stable towers of order $n$ are there, and does the probability $x_n$ of a tower of order $n$ being stable decrease to zero as $n$ increases?

(4) What number, $m_n$ of maximally stable towers or order $n$ are there?

1. The following recursion relation can be seen immediately by taking moments about the point A in Figure 7.

$$T_{n+1} = \sum_{i=0}^{n} T_{n-i} + 1.$$



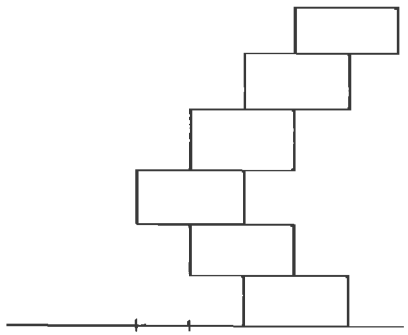



Figure 7 Figure 8

where the final 1 is required to ensure stability, (as opposed to unstable equilibrium).

The solution of this is:

$$T_0 = 1,$$

$$T_n = 2^n \quad n \geqslant 1.$$

2. If we permit towers of the form shown in Figure 8 the solution is much more economical for large $n$. The center of mass of the upper part of the tower, $m$ bricks high is $m + 1/2$ units from the left. Hence for balance we must add $m - 1/2$ bricks to the right. Hence the overhang is

$$n = m - 1 - \frac{m-1}{2} \quad \text{for a total of} \quad T_n = m + \frac{m-1}{2}$$

i.e. $$n = \frac{T_n + 1}{3} \quad (T_n = 3K - 1)$$

$$n = \frac{T_n - 1}{3} \quad (T_n = 3K + 1)$$

giving the solution

$$T_n = 3n - 1$$

which is more economical than the above solution for $n > 3$.

3, 4. There seems to be no simple solution, but the problem is easily set up for computer solution.



If we label the first block as in Figure 9 then position of the center of the second block will be at $P_2 = -1, 0$ or $1$ and the corresponding center of mass at $c_2 = -\frac{1}{2}, 0$ or $\frac{1}{2}$. This immediately generalizes to

$$p_n = p_{n-1} + k_n \qquad (1)$$

where $k_n = -1, 0$ or $1$, and

$$n\, c_n = (n-1)\, c_{n-1} + p_n \qquad (2)$$

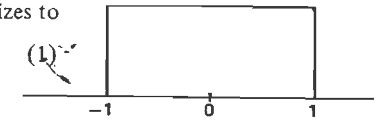



Figure 9

The solutions are then obtained by the somewhat inelegant method of constructing all possible towers from (1) and finding their center of mass $c_n$ from (2). The solution to problem (3) requires that $c_n < 1$, while that for (4) requires $c_n = 0$. We find the following results:

| $n$ | $Sn$ | $Xn$ | $Mn$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 3 | 1 | 1 |
| 3 | 7 | .778 | 1 |
| 4 | 19 | .704 | 3 |
| 5 | 53 | .654 | 7 |
| 6 | 149 | .613 | 15 |
| 7 | 419 | .575 | 35 |
| 8 | 1191 | .544 | 87 |
| 9 | 3403 | .518 | 217 |
| 10 | 9755 | .495 | 547 |
| 11 | 28077 | .475 | 1417 |
| 12 | 81097 | .457 | 3735 |

The proportion of stable towers $X_n = Sn/3n$ decreases roughly as

$$x_n \sim \frac{1}{\log_3 n},$$

while the proportion of maximally stable towers seems to go as

$$M_n \sim \frac{1}{n \log_3 n}.$$

It therefore seems probable that as $n$ goes to infinity, there are infinitely many stable towers, but the probability of any random tower being stable tends to zero.

I am grateful to Mr. David Stanley for the gift of the Lego.