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Knuth

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Aug 30 19:51 1994   Page 1

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From CS.Stanford.EDU!winkler Tue Aug 30 13:12:26 0700 1994
Received: by research.att.com; Tue Aug 30 16:12 EDT 1994
Received: by Sunburn.Stanford.EDU (5.67b/25-SUNBURN-eef) id AA01095; Tue, 30 Aug 1994 13:12:2
Date: Tue, 30 Aug 1994 13:12:26 -0700
From: Phyllis Winkler <winkler@CS.Stanford.EDU>
Message-Id: <199408302012.AA01095@Sunburn.Stanford.EDU>
To: njas@research.att.com
Subject: note from Don Knuth
Reply-To: winkler@CS.Stanford.EDU
Status: RO

Dear njas,

Here's the sequence that will be published at least in part some day in The Art of Computer Programming, God willing, in the solution to one of the exercises in Section 7.3.

I also append a Mathematica program to compute it. The next generation of your Encyclopedia will, of course, include online algorithms for all sequences, right?

$x_0 \ldots x_n$

The problem is to count all sequences $(x_0,\ldots,x_n)$ so that there is a Sperner family = clutter = antichain = family of incomparable subsets of $\{1,\ldots,n\}$ having x_k members of cardinality k. A characterization of such $(x_0,\ldots,x_n)$ was found by Clements, Discrete Mathematics 3 (1973) 123--128; in fact, he solved a more general problem about antichains of multisets instead of sets.

$1 \ldots n$

Let $f(n)$ be the number of feasible $(x_0,\ldots,x_n)$. Then $f(1),\ldots,f(16)$ are:

3
5
10
26
96
553
5461
100709
3718354
289725509
49513793526
19089032278261
16951604697397302
35231087224279091310
173550485517380958360611
2047581288200721764035942914

And here is the (inefficient) program I used:

```
c[0,0]=1
c[0,1]=1
kap[0,0]=0
f[n_]:=Block[{s=2,r,d,k,j},
 For[r=1,r<=n,r++,
  d=s; k=r; j=0; s=0;
```

```
  For[x=0, x<=Binomial[n,r], x++,
    If[x>=Binomial[k,r],k++,0];
    kap[r,x]=If[x==0,0,Binomial[k-1,r-1]+kap[r-1,x-Binomial[k-1,r]]];
    While[j<kap[r,x], d -= c[r-1,j];j++];
    c[r,x]=d;
    s += d;
  ]
 ];
 s
]
```

PS: Another sequence you ought to have, if you don't already, is $S_n=\sum_{k=0}^n {2k\choose k}$. This one arose in connection with another exercise I thought of Sunday morning, but I'm sure it has popped up elsewhere. The exercise, which is related to a beautiful theorem of Joe Kruskal from the early 60s, is this: Find the smallest n such that there's a family of n sets of cardinality r having fewer than n sets in its shadow. (The ``shadow'' of a family is the family of all sets obtained by deleting one element. For example, the shadow of $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,4\}$ is $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$. When $r=2$ the answer to the question is obviously 5. In general the answer is $(1+S_r)/2$. Here is a table of S_n for $0\le n\le 20$:

```
{1, 3, 9, 29, 99, 351, 1275, 4707, 17577, 66197, 250953, 956385,
   3660541, 14061141, 54177741, 209295261, 810375651, 3143981871,
   12219117171, 47564380971, 185410909791}
```

I'm sure you don't want the partial sums of every sequence to be included, but the number of times I look up first differences in your table is probably less than it should be and I suspect the same is true for other users.