Unitarism and Infinitarism

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We will examine variations of four famous arithmetical functions. For a given function χ , let χ^* denote its unitary analog, $\tilde{\chi}$ its square-free analog, and χ' its unitary square-free analog. The meanings of these phrases will be made clear in each case. At the end, the infinitary analog χ_{∞} will appear as well.

0.1. Divisor Function. If d(n) is the number of distinct divisors of n, then

$$\sum_{n=1}^{N} d(n) = N \ln(N) + (2\gamma - 1)N + O(\sqrt{N})$$

as $N \to \infty$, where γ is the Euler-Mascheroni constant. Let us introduce a more refined notion of divisibility. A divisor k of n is **unitary** if k and n/k are coprime, that is, if $\gcd(k, n/k) = 1$. This condition is often written as k||n. The number $d^*(n)$ of unitary divisors of n is $2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. This fact is easily seen to be true: If $p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ is the prime factorization of n, then the unitary divisors of n are of the form $p_1^{\varepsilon_1 a_1}p_2^{\varepsilon_2 a_2}\cdots p_r^{\varepsilon_r a_r}$, where each ε_s is either 0 or 1. There are 2^r possible choices for the r-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$; hence the result follows. We have [1, 2, 3, 4, 5]

$$\sum_{n=1}^{N} d^*(n) = \frac{6}{\pi^2} N \ln(N) + \frac{6}{\pi^2} \left(2\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) \right) N + O(\sqrt{N}),$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta'(x)$ is its derivative.

A divisor k of n is **square-free** if k is divisible by no square exceeding 1. The number $\tilde{d}(n)$ of square-free divisors of n is also $2^{\omega(n)}$; the divisors in this case are of the form $p_1^{\varepsilon_1}p_2^{\varepsilon_2}\cdots p_r^{\varepsilon_r}$. Therefore the same asymptotics apply for $\tilde{d}(n)$, but the underlying sets of numbers overlap only somewhat [6].

Define d'(n) to be the number of unitary square-free divisors of n. A more complicated asymptotic formula arises here [7, 8]:

$$\sum_{n=1}^{N} d'(n) = \frac{6\alpha}{\pi^2} N \ln(N) + \frac{6\alpha}{\pi^2} \left(2\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) + X \right) N + O(\sqrt{N} \ln(N))$$

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where

$$\alpha = \prod_{p} \left(1 - \frac{1}{p(p+1)} \right) = 0.7044422009..., \quad X = \sum_{p} \frac{(2p+1)\ln(p)}{(p+1)(p^2+p-1)}$$

and we agree that the product and sum extend over all primes p. The constant α is the same as what is called $\pi^2 P/6$ in [9].

We finally give corresponding reciprocal sums [10, 11, 12]:

$$\lim_{N \to \infty} \frac{\sqrt{\ln(N)}}{N} \sum_{n=1}^{N} \frac{1}{d(n)} = \frac{1}{\sqrt{\pi}} \prod_{p} \sqrt{p(p-1)} \ln\left(\frac{p}{p-1}\right) = \frac{0.9692769438...}{\sqrt{\pi}}$$

$$\lim_{N \to \infty} \frac{\sqrt{\ln(N)}}{N} \sum_{n=1}^{N} \frac{1}{d^*(n)} = \frac{1}{\sqrt{\pi}} \prod_{n} \sqrt{1 + \frac{1}{4p(p-1)}} = \frac{1.0969831191...}{\sqrt{\pi}}$$

The former sum was mentioned in [13] with regard to the arcsine law for random divisors. It is not known what constant emerges for 1/d'(n).

0.2. Sum-of-Divisors Function. If $\sigma(n)$ is the sum of all distinct divisors of n, then

$$\sum_{n=1}^{N} \sigma(n) = \frac{\pi^2}{12} N^2 + O(N \ln(N))$$

as $N \to \infty$. Let $\sigma^*(n)$ be the sum of unitary divisors of n and $\tilde{\sigma}(n)$ be the sum of square-free divisors of n. Although $d^*(n) = \tilde{d}(n)$ always, it is usually false that $\sigma^*(n) = \tilde{\sigma}(n)$ [14]. We have [15, 16, 17, 18]

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \sigma^*(n) = \frac{\pi^2}{12\zeta(3)}, \qquad \lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \tilde{\sigma}(n) = \frac{1}{2}.$$

Further, if $\sigma'(n)$ is the sum of unitary square-free divisors of n, then [15]

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \sigma'(n) = \frac{1}{2} \prod_{p} \left(1 - \frac{1}{p^2(p+1)} \right) = \frac{0.8815138397...}{2},$$

a constant which appeared in [19] and turns out to be connected with class number theory [20, 21, 22].

Corresponding reciprocal sums are [23, 24]

$$\sum_{n=1}^{N} \frac{1}{\sigma(n)} \sim Y_1 \ln(N) + Y_1(\gamma + Y_2), \qquad \sum_{n=1}^{N} \frac{1}{\sigma^*(n)} \sim Y_3 \ln(N) + Y_3(\gamma + Y_4 - Y_5)$$

where

$$Y_{1} = \prod_{p} f(p), \qquad Y_{2} = \sum_{p} \frac{(p-1)^{2} g(p) \ln(p)}{p f(p)},$$

$$Y_{3} = \prod_{p} \left(1 - \frac{p}{p-1} \sum_{j=1}^{\infty} \frac{1}{p^{j}(p^{j}+1)} \right), \qquad Y_{4} = \sum_{p} \left(\frac{p h(p) \ln(p)}{p-1} \sum_{j=1}^{\infty} \frac{j}{p^{j}(p^{j+1}+1)} \right),$$

$$Y_{5} = \sum_{p} \left(\frac{h(p) \ln(p)}{p^{2}} \sum_{j=0}^{\infty} \frac{1}{p^{j}(p^{j+1}+1)} \right), \qquad f(p) = 1 - \frac{(p-1)^{2}}{p} \sum_{j=1}^{\infty} \frac{1}{(p^{j}-1)(p^{j+1}-1)},$$

$$g(p) = \sum_{j=1}^{\infty} \frac{j}{(p^{j}-1)(p^{j+1}-1)}, \qquad h(p) = 1 - \frac{p}{p-1} \sum_{j=1}^{\infty} \frac{1}{p^{j}(p^{j+1}+1)}.$$

No one seems to have examined $1/\tilde{\sigma}(n)$ or $1/\sigma'(n)$ yet.

0.3. Totient Function. If $\varphi(n)$ is the number of positive integers $k \leq n$ satisfying gcd(k, n) = 1, then [25, 26]

$$\sum_{n=1}^{N} \varphi(n) = \frac{3}{\pi^2} N^2 + O(N \ln(N))$$

as $N \to \infty$. Define $\gcd_*(k, n)$ to be the greatest divisor of k that is also a unitary divisor of n. Let $\varphi^*(n)$ be the number of positive integers $k \le n$ satisfying $\gcd_*(k, n) = 1$. Since \gcd_* is never larger than \gcd_* it follows that φ^* is at least as large as φ . Also let $\tilde{\varphi}(n)$ be the number of positive square-free integers $k \le n$ satisfying $\gcd(k, n) = 1$. We have [15, 27]

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \varphi^*(n) = \frac{1}{2} \alpha, \qquad \lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \tilde{\varphi}(n) = \frac{3}{\pi^2} \alpha$$

where α is as defined earlier. The case for $\varphi'(n)$ remains open.

Corresponding reciprocal sums are [23, 24, 28]

$$\sum_{n=1}^{N} \frac{1}{\varphi(n)} \sim Z_1 \ln(N) + Z_1(\gamma - Z_2), \qquad \sum_{n=1}^{N} \frac{1}{\varphi^*(n)} \sim Z_3 \ln(N) + Z_3(\gamma - Z_4 + Z_5 + Z_6)$$

where

$$Z_1 = \frac{315\zeta(3)}{2\pi^4}, \qquad Z_2 = \sum_p \frac{\ln(p)}{p^2 - p + 1}, \qquad Z_3 = \prod_p u(p),$$

$$Z_4 = \sum_{p} \left(\frac{(p-1)\ln(p)}{p u(p)} \sum_{j=1}^{\infty} \frac{j}{p^j(p^j-1)} \right),$$

$$Z_5 = \sum_{p} \frac{\ln(p)}{p^2(p-1)u(p)}, \quad Z_6 = \sum_{p} \frac{v(p)\ln(p)}{p^2 u(p)},$$

$$u(p) = 1 + \frac{p-1}{p} \sum_{j=1}^{\infty} \frac{1}{p^j(p^j-1)}, \quad v(p) = \sum_{j=1}^{\infty} \frac{1}{p^j(p^{j+1}-1)}.$$

0.4. Square-Free Core Function. If $\tilde{\kappa}(n)$ is the maximal square-free divisor of n (also called [9] the square-free kernel of n), then [15, 17, 18, 29, 30, 31]

$$\sum_{n=1}^{N} \tilde{\kappa}(n) = \frac{\alpha}{2} N^2 + O\left(N^{3/2}\right)$$

as $N \to \infty$, where α is as before. Assuming the Riemann hypothesis, the error term can be improved to $O(N^{7/5+\varepsilon})$ for any $\varepsilon > 0$. If $\kappa'(n)$ is the maximal unitary square-free divisor of n, then [30, 31]

$$\sum_{n=1}^{N} \kappa'(n) = \frac{\beta}{2} N^2 + O\left(N^{3/2}\right)$$

where

$$\beta = \prod_{p} \left(1 - \frac{p^2 + p - 1}{p^3(p+1)} \right) = 0.6496066993....$$

0.5. Infinitary Arithmetic. We continue refining the notion of divisibility [32, 33]. A divisor k of n is biunitary if the greatest common unitary divisor of k and n/k is 1, and triunitary if the greatest common biunitary divisor of k and n/k is 1. More generally, for any positive integer m, a divisor k of n is m-ary if the greatest common (m-1)-ary divisor of k and n/k is 1. We write $k|_{m}n$. Clearly $1|_{m}n$ and $n|_{m}n$.

When introducing infinitary divisors, it is best to start with prime powers. Let p be a prime, and let $x \ge 0$, $y \ge 1$ be integers. It can be proved that, for any $m \ge y-1$, $p^x|_m p^y$ if and only if $p^x|_{y-1} p^y$. Thus we define $p^x|_{\infty} p^y$ if $p^x|_{y-1} p^y$. For fixed y, the number of integers $0 \le x \le y$ satisfying $p^x|_{\infty} p^y$ is $2^{b(y)}$, where b(y) is the number of ones in the binary expansion of y. Define as well $1|_{\infty}1$. The sum $\sum_{y=0}^{z-1} 2^{b(y)}$ is approximately $z^{\ln(3)/\ln(2)}$ but is not well behaved asymptotically [34].

We now allow n to be arbitrary. A divisor k of n is **infinitary** if, for any prime p, the conditions $p^x||k$ and $p^y||n$ imply that $p^x|_{\infty}p^y$. We write $k|_{\infty}n$. Clearly $1|_{\infty}n$ and

 $n|_{\infty}n$. Each n>1 has a unique factorization as a product of distinct elements from the set

 $I = \left\{ p^{2^j} : p \text{ is prime and } j \ge 0 \right\};$

each element of I in this product is called an I-component of n. It follows that $k|_{\infty}n$ if and only if every I-component of k is also an I-component of n.

Assume that $n = P_1 P_2 \cdots P_t$, where $P_1 < P_2 < \cdots < P_t$ are the *I*-components of n. The infinitary analogs of the functions d and σ are defined by [35, 36]

$$d_{\infty}(n) = 2^t, \quad \sigma_{\infty}(n) = \prod_{i=1}^t (P_i + 1),$$

for n > 1; otherwise $d_{\infty}(1) = \sigma_{\infty}(1) = 1$. Two infinitary analogs of the function φ are known:

 $\varphi_{\infty}(n) = \text{ the number of positive integers } k \leq n \text{ satisfying } \gcd_{\infty}(k, n) = 1;$

$$\hat{\varphi}_{\infty}(n) = \prod_{i=1}^{t} (P_i - 1) = n \prod_{i=1}^{t} \left(1 - \frac{1}{P_i} \right) \text{ for } n > 1, \quad \hat{\varphi}_{\infty}(1) = 1.$$

It is generally untrue that $\varphi_{\infty}(n) = \hat{\varphi}_{\infty}(n)$. No similar extension of the function $\tilde{\kappa}$ is known. Cohen & Hagis [35, 37] proved that

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \sigma_{\infty}(n) = \frac{A}{2} = 0.7307182421...,$$

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \hat{\varphi}_{\infty}(n) = \frac{B}{2} = 0.3289358388...,$$

$$\frac{1}{N^2} \sum_{n=1}^{N} d_{\infty}(n) \sim CN \ln(N) + DN \sim 2(0.3666252769...) N \ln(N)$$

where

$$A = \prod_{P \in I} \left(1 + \frac{1}{P(P+1)} \right), \quad B = \prod_{P \in I} \left(1 - \frac{1}{P(P+1)} \right), \quad C = \prod_{P \in I} \left(1 - \frac{1}{(P+1)^2} \right)$$

but no such expression for D yet exists. It is known that $\varphi_{\infty}(n) = n^2/\sigma_{\infty}(n) + O(n^{\varepsilon})$ for any $\varepsilon > 0$; reciprocal sums involving d_{∞} , σ_{∞} and $\hat{\varphi}_{\infty}$ also remain open. Alternative generalizations of unitary divisor have been given [38, 39] but won't be discussed here.

0.6. Addendum. The constant X, which is associated with counting unitary square-free divisors of n, was evaluated in [40] to be 0.7483723334.... The series $\sum_{n\leq N} 1/d(n)$ connected with $(0.5468559552...)N \ln(N)^{-1/2}$ was mentioned in [41, 42], as well as a numerical value 0.6728... for Y_1 . Other values of Y_i , Z_j remain open. An analog of d'(n), corresponding to unitary cube-free divisors of n, can be studied [8, 40]. The probability that k randomly chosen integers are unitary coprime is [43]

$$\prod_{p} \left(1 - \frac{(p-1)^k}{p^k(p^k - 1)} \right).$$

The probability that they are *pairwise* unitary coprime is more complicated: for instance, it is

$$\zeta(2)\zeta(3)\prod_{p}\left(1-\frac{4}{p^2}+\frac{7}{p^3}-\frac{9}{p^4}+\frac{8}{p^5}-\frac{2}{p^6}-\frac{3}{p^7}+\frac{2}{p^8}\right)$$

when k = 3 and

$$\zeta(2)^2 \zeta(3) \zeta(4) \prod_p \left(1 - \frac{8}{p^2} + \frac{3}{p^3} + \frac{27}{p^4} - \frac{24}{p^5} - \frac{14}{p^6} - \frac{3}{p^7} + \frac{37}{p^8} - \frac{30}{p^9} + \frac{42}{p^{10}} - \frac{33}{p^{11}} - \frac{41}{p^{12}} + \frac{78}{p^{13}} - \frac{44}{p^{14}} + \frac{9}{p^{15}} \right)$$

when k = 4. Expressions for arbitrary k appear in [43].

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