On the circle and divisor problems

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Abstract

We claim that the Dirichlet divisor problem and the Gauss circle problem are equivalent. To do this we conjecture a deep property of the fractional part function $x \mapsto \{x\}$ reminiscent of our tauberian approach to the Riemann hypothesis [Clo] where we consider the related function $x \mapsto x^{-1} \lfloor x \rfloor$.

To the memory of Herbert Wilf.

Introduction

The Dirichlet divisor problem

The Dirichlet divisor problem (DDP for short) consists to find the sharpest values of θ_0 such that we have

$$D(n) = \sum_{1 \leq k \leq n} \tau(k) = n \log n + (2\gamma - 1)n + O(n^{\theta_0}) \tag{1}$$

where $\tau(k)$ counts the number of divisors of k and γ is the Euler gamma constant. This can be rewritten using the floor function:

$$D(n) = \sum_{1 \leq k \leq n} \left\lfloor \frac{n}{k} \right\rfloor \tag{2}$$

As we shall see the formulation (2) is the best one for our purpose. It is widely believed that $\theta_0 = \frac{1}{4} + \varepsilon$ for any $\varepsilon > 0$ is working. Since the earliest work of Dirichlet who showed using the ingenious hyperbola method that we can take $\theta_0 = \frac{1}{2}$ few progresses were made. In 1905 Voronoï improved significantly the bound to $\theta_0 = \frac{1}{3} + \varepsilon$ and Kolesnik in 1982 found the slightly better $\theta_0 = 0.324074...$ [Ivi] and there is continuous work on the subject [Cha] until recently [BBR]. The best known result is due to Huxley in 2003 [Hux] with $\theta_0 = \frac{131}{416} = 0.31490...$ However we are still far from $\frac{1}{4}$.

The Gauss circle problem

The Gauss circle problem (GCP for short) is related to the counting function

$$G(n) = \left|(x,y) \in \mathbb{Z}^2 \;:\; x^2+y^2 \leq n^2\right| = \sum_{1\leq k\leq n^2} r_2(k) = \pi n^2 + O(n^{2\theta_0^\star}) \qquad (3)$$

where $r_2(k)$ denotes the number of ways to write k as a sum of 2 squares. In order to introduce similarities with the DDP we prefer to use the following formula due to Gauss and involving the floor function

$$g(n) := G(\sqrt{n}) - 1 = 4\sum_{0\leq k\leq \frac{n}{4}} \left\lfloor \frac{n}{4k+1} \right\rfloor - 4\sum_{0\leq k\leq \frac{n}{4}} \left\lfloor \frac{n}{4k+3} \right\rfloor$$

so that we have

$$g(n) = \pi n + O(n^{\theta_0^\star}) \qquad (4)$$

and it is conjectured that $\theta_0^\star = \frac{1}{4} + \varepsilon$ is again the best possible choice. The best known result to this date is again due to Huxley in 2003 [Hux] with the same value as above $\theta_0^\star = \frac{131}{416} = 0.31490....$

Analogies between the 2 problems

Many people feel both problems are related and there are nice surveys about DDP and GCP and general lattice points counting [IKK, Cha].

A trivial geometric analogy is that DDP counts lattice points under an hyperbola and GCP counts lattice points within a circle.

Richert already transformed the circle problem into a divisor problem from an analytic view point (1957) [Ivi] and Ivic added in his book [Ivi]:

"*The approach presented here shows a unified view of the circle and divisor problem*" .

In [Miy] the author provides 2 striking formulas showing another clear analytic similitude between DDP and GCP. But before that Ramanujan himself stated amazing similar formulas for both problems [Ram][Ber].

Goal of the paper

In this article we aim to show the problems are in fact equivalent and the consequence of a deep conjectural property of the fractional part function. We state this main conjecture in section 1.

In section 2 we rewrite DDP and CGP in order to see they fit the main conjecture.

In section 3 we provide experimental support of the main conjecture using a conjectural trick based on low discrepancy sequences.

Finally in section 4 we discuss generalisation of the main conjecture and an analogy with our good variation concept [Clo].

1 Main conjecture

This is the main conjecture since it encapsulates both DDP and GCP as we will see in section 2. Here we consider the fractional part function $x \mapsto \{x\}$ and for a given smooth function f Riemann integrable over $[0,1]$, for 2 fixed real values $\lambda \geq 1, \mu \geq 0$, we define the integral

$$I_f := \int_0^1 f\left(\left\{\frac{1}{t}\right\}\right) dt$$

and the corresponding family of Riemann sums

$$S_f(N,\lambda,\mu) := \lambda \sum_{1 \leq k \leq \frac{N}{\lambda}} f\left(\left\{\frac{N}{\lambda k + \mu}\right\}\right)$$

So that we have

$$\lim_{N\to\infty} \frac{S_f(N,\lambda,\mu)}{N} = \lambda \int_0^{1/\lambda} f\left(\left\{\frac{1}{\lambda t}\right\}\right) dt = \int_0^1 f\left(\left\{\frac{1}{u}\right\}\right) du = I_f$$

Then we make the following conjecture relating I_f to $S_f(N,\lambda,\mu)$. This conjecture tells us that the fractional part function has a very deep intrinsinc property. We provide a weak form, a stronger from and the strongest form we believe to be true so far.

1.1 Main conjecture

Weak form

Suppose $f(x) = x$ then for any $\lambda \geq 1, \mu \geq 0$ we have $\forall \varepsilon > 0$

$$S_f(N,\lambda,\mu) = (1-\gamma)N + O\left(N^{\frac{1}{4}+\varepsilon}\right)$$

Strong form

Suppose f is continuous on the interval $[0,1]$ and of bounded variation, then for any $\lambda \geq 1, \mu \geq 0$ we have $\forall \varepsilon > 0$

$$S_f(N,\lambda,\mu) = I_f N + O\left(N^{\frac{1}{4}+\varepsilon}\right)$$

More precisely we claim there is a slowly varying function L_f [1]such that we have

$$S_f(N,\lambda,\mu) = I_f N + O\left(N^{\frac{1}{4}} L_f(N)\right)$$

And for $f(x) = x$ we can take $L_f(x) = \log x$.

[1] L is a slowly varying function if for any $x > 0$ we have $\lim_{t\to\infty} \frac{L(xt)}{L(t)} = 1$.

Strongest form

In fact we state something more general. Suppose f and g are continuous on the interval $[0, 1]$ and define

$$I_{f,g}(\nu) := \int_0^{\nu} f\left(\left\{\frac{1}{t}\right\}\right) g(t)dt$$

Let

$$S_{f,g}(N, \lambda, \mu, \nu) := \lambda \sum_{1 \le k \le \frac{N}{\lambda}} f\left(\left\{\frac{\nu N}{\lambda k + \mu}\right\}\right) g\left(\frac{\lambda k + \mu}{\nu N}\right)$$

Then for any $\lambda \ge 1, \mu \ge 0$ and $0 < \nu \le 1$ we have

$$S_{f,g}(N, \lambda, \mu, \nu) = I_{f,g}(\nu)N + O\left(N^{\frac{1}{4}} L_{f,g}(N)\right)$$

where $L_{f,g}$ is a slowly varying function.

Although we don't provide directly experimental support for this conjecture in this draft we provide some examples in 2.3.

2 GCP and DDP fit the the main conjecture

We prove that GCP and the DDP fit the weak form of the main conjecture described in 1.1.. We begin by DDP for practical and not historical reasons.

2.1 DDP

2.1.1 Theorem 2.1.1

We have

$$T(n) := \sum_{1 \le k \le n} \left\{\frac{n}{k}\right\} = (1 - \gamma)n + O(n^{\theta_0})$$

where θ_0 is the value considered in the formula (1).

Proof Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ then we have

$$\sum_{1 \le k \le n} \left\{\frac{n}{k}\right\} = \sum_{1 \le k \le n} \frac{n}{k} - \left\lfloor \frac{n}{k} \right\rfloor = nH_n - D(n)$$

Next it is known that $H_n = \log n + \gamma + O(n^{-1})$ hence using the asymptotic formula (1) we get

$$T(n) = (1 - \gamma)n + O(n^{\theta_0})$$

2.1.2 The DDP fits the main conjecture

Indeed it is well known we have

$$\int_0^1 \left\{\frac{1}{t}\right\} dt = 1 - \gamma$$

hence letting $f(x) = x, \lambda = 1, \mu = 0$ we have from the main conjecture and theorem 2.1.1.

$$S_f(N, \lambda, \mu) = T(n) = (1-\gamma)N + O\left(N^{\frac{1}{4}+\varepsilon}\right)$$

implying that $\theta_0 = \frac{1}{4} + \varepsilon$ is the best possible choice for the DDP.

2.2 GCP

This is slightly more complicated and we need to split the problem in 2 sums. We consider $g(n)$ given in the formula (4)

$$g(n) = 4\sum_{k\geq 0}\left\lfloor \frac{n}{4k+1} \right\rfloor - 4\sum_{k\geq 0}\left\lfloor \frac{n}{4k+3} \right\rfloor$$

Then we define

$$S_1(n) := 4\sum_{k\geq 0}\left\lfloor \frac{n}{4k+1} \right\rfloor$$

and

$$S_3(n) := 4\sum_{k\geq 0}\left\lfloor \frac{n}{4k+3} \right\rfloor$$

and we show these 2 sums are divisors problems like via the following theorem.

2.2.1 Theorem 2.2.1

Let $\theta_1 = \max(\theta_0, \theta_0^\star)$ where $\theta_0^\star$ is the value considered in the asymptotic formula (4). Then we have

$$S_1(n) = n\log n + \left(2\gamma - 1 + \log 2 + \frac{\pi}{2}\right) n + O(n^{\theta_1})$$

$$S_3(n) = n\log n + \left(2\gamma - 1 + \log 2 - \frac{\pi}{2}\right) n + O(n^{\theta_1})$$

and we have also

$$T_1(n) := 4\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor}\left\{ \frac{n}{4k+1} \right\} = (1-\gamma)n + O\left(n^{\theta_1}\right)$$

$$T_3(n) := 4\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor}\left\{ \frac{n}{4k+3} \right\} = (1-\gamma)n + O\left(n^{\theta_1}\right)$$

Proof of theorem 2.2.1. We have

$$\frac{S_1(2n)+S_3(2n)}{4} = D(2n) - D(n)$$

Hence from the asymptotic formula (2) we get

$$S_1(n)+S_3(n) = 2n\log(n) + 2(2\gamma - 1 + \log 2)n + O(n^{\theta_0})$$

And from the asymptotic formula (4) we have

$$S_1(n) - S_3(n) = \pi n + O(n^{\theta_0^\star})$$

Thus letting $\theta_1 = \max(\theta_0, \theta_0^\star)$ and combining the 2 previous formulas we have

$$S_1(n) = n\log n + \left(2\gamma - 1 + \log 2 + \frac{\pi}{2}\right)n + O(n^{\theta_1})$$

$$S_3(n) = n\log n + \left(2\gamma - 1 + \log 2 - \frac{\pi}{2}\right)n + O(n^{\theta_1})$$

In an other hand we have by definition of T_1 and T_3

$$T_1(n) = n\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{k+\frac{1}{4}} - S_1(n)$$

$$T_3(n) = n\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{k+\frac{3}{4}} - S_3(n)$$

And we have (details omitted)

- $\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{k+\frac{1}{4}} = \log(n) - 2\log 2 - \psi\left(\frac{1}{4}\right) + O(n^{-1})$
- $\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{k+\frac{3}{4}} = \log(n) - 2\log 2 - \psi\left(\frac{3}{4}\right) + O(n^{-1})$

where $\psi = \frac{\Gamma'}{\Gamma}$ is the digamma function yielding

- $\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - \gamma - 3\log 2$
- $\psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - \gamma - 3\log 2$

Thus we have some simplifications and we get finally

$$T_1(n) = (1-\gamma)n + O(n^{\theta_1})$$

$$T_3(n) = (1-\gamma)n + O(n^{\theta_1})$$

proving the theorem 2.2.1.

2.2.2 The GCP fits the main conjecture

Indeed from theorem 2.2.1 we have

$$\lim_{n\to\infty}\frac{1}{n}T_1(n)=\lim_{n\to\infty}\frac{1}{n}T_3(n)=4\int_0^{1/4}\left\{\frac{1}{4t}\right\}dt=\int_0^1\left\{\frac{1}{u}\right\}du=1-\gamma$$

Hence choosing $f(x)=x,\lambda=4,\mu=1$ we have from the main conjecture and the theorem 2.2.1.

$$S_f(n,\lambda,\mu)=T_1(n)=(1-\gamma)n+O\left(n^{1/4+\varepsilon}\right)\Rightarrow\theta_1=\frac{1}{4}+\varepsilon$$

or equivalently choosing $f(x)=x,\lambda=4,\mu=3$ we have the same thing from the main conjecture and the theorem 2.2.1.

$$S_f(n,\lambda,\mu)=T_3(n)=(1-\gamma)n+O\left(n^{1/4+\varepsilon}\right)\Rightarrow\theta_1=\frac{1}{4}+\varepsilon$$

Hence we have necessarily

$$g(n)=S_1(n)-S_3(n)=\pi n+O(n^{1/4+\varepsilon})$$

implying that $\theta_0^\star=\frac{1}{4}+\varepsilon$ is the best possible choice for the GCP.

2.3 Other examples

Many other lattice couting problems or related problems fit our main conjecture 2.1.. We provide some additional examples, the 2 last ones being examples of the strongest form of the conjecture.

2.3.1 Another circle problem

This is an interesting example of lattice points counting. Let us consider the hexagonal lattice and the number of lattice points which are within a circle of radius n centered at the origin, i.e. that is

$$h(n)=\#\left\{(x,y)\in\mathbb{Z}^2\ :\ x^2+xy+y^2\leq n\right\}$$

which is the sequence A038589. Then we have

$$h(n)=\frac{2\pi}{\sqrt{3}}n+O(n^{\theta_2})$$

where we conjecture

$$\theta_2=\frac{1}{4}+\varepsilon$$

is again the best admissible choice. It is indeed easy to show that this problem fits also the main conjecture 1.1. due to the other formula

$$h(n) = 1 + 6 \sum_{k=1}^{\lfloor n/3 \rfloor} \left(\left\lfloor \frac{n}{3k+1} \right\rfloor - \left\lfloor \frac{n}{3k+2} \right\rfloor \right)$$

(details ommitted) and using the same method than for proving GCP fits the weak form of the main conjecture in 2.2..

2.3.2 On sums involving the floor function

We support somewhat the strongest form of the main conjecture providing the following examples.

First example We consider the sum

$$S(n) := \sum_{k \geq 1} \left\lfloor \frac{n}{k^2} \right\rfloor$$

which is in fact the sequence A013936 and there is this asymptotic formula

$$S(n) = \zeta(2)n + \zeta\left(\frac{1}{2}\right) n^{1/2} + O\left(n^{\theta_3}\right) \tag{5}$$

where we claim we can take again $\theta_3 = \frac{1}{4} + \varepsilon$. Indeed it is easy to see this sum can be rewritten as follows

$$S(n) = \sum_{k \geq 1} \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor$$

So that we have to consider from our view point of DDP and GCP

$$U(n) = \sum_{k=1}^{n} \left\{ \sqrt{\frac{n}{k}} \right\} = n^{1/2} \sum_{k=1}^{n} k^{-1/2} - S(n)$$

Next we have (details omitted)

$$n^{1/2} \sum_{k=1}^{n} k^{-1/2} = 2n + \zeta\left(\frac{1}{2}\right) n^{1/2} + \frac{1}{2} + o(1)$$

Hence we get using (5)

$$U(n) = (2 - \zeta(2))n + O\left(n^{\theta_3}\right)$$

And in an other hand it is easy to see

$$I = \int_0^1 \left\{ \sqrt{\frac{1}{t}} \right\} dt = 2 - \zeta(2)$$

Therefore the variable change $t = u^2$ yields

$$I = \int_0^1 \left\{ \frac{1}{u} \right\} (2u) du = 2 - \zeta(2)$$

Thus the strongest form of the conjecture 2.1. is working and tells us that choosing

- $f(x) = x, g(x) = 2x, \lambda = 1, \mu = 0, \nu = 1$

we have

$$S_{f,g}(N, \lambda, \mu, \nu) = U(n) = (2 - \zeta(2))n + O(n^{1/4+\varepsilon})$$

And then $\theta_3 = \frac{1}{4} + \varepsilon$ is the sharpest value for this problem too.

Second example Namely we consider

$$S(n) := \sum_{k \geq 1} \left\lfloor \frac{n^2}{k^2} \right\rfloor$$

which is sequence A153818. Then we claim we have

$$S(n) = \zeta(2)n^2 + \zeta\left(\frac{1}{2}\right)n + O\left(n^{1/4+\varepsilon}\right) \tag{6}$$

Indeed we have

$$I = \int_0^1 \left\{ \frac{1}{t^2} \right\} dt = -1 - \zeta\left(\frac{1}{2}\right)$$

then making the variable change $t = u^{1/2}$ we get

$$I = \int_0^1 \left\{ \frac{1}{u} \right\} \left(\frac{u^{-1/2}}{2} \right) du = -1 - \zeta\left(\frac{1}{2}\right)$$

Hence the strongest form of the conjecture 2.1. tells us that choosing

- $f(x) = x, g(x) = \frac{x^{-1/2}}{2}, \lambda = 1, \mu = 0, \nu = 1$

we have

$$S_{f,g}(N, \lambda, \mu, \nu) = \left(-1 - \zeta\left(\frac{1}{2}\right)\right) n + O(n^{1/4+\varepsilon})$$

yielding (6).

3 Experimental support

At first glance it is very hard to check directly the main conjecture for 2 reasons:

1. the value of $\int_0^1 f\left(\left\{\frac{1}{t}\right\}\right) dt$ is in general difficult to find.
2. The Riemann sums converge slowly and we need to compute the sum for each n.

and we didn't try to find general like Voronoï formula for the Riemann sums.

Regarding 1.

In order to check several functions f we succeeded to find an efficient formula when $f(x) = x^\alpha$. Namely we have for $\alpha \notin \{0,1\} \cup \{-k \ : \ k \in \mathbb{N}\}$

$$\int_0^1 \left\{\frac{1}{t}\right\}^\alpha dt = \frac{\alpha}{\alpha-1} - \frac{\alpha}{2}\left(\psi\left(\frac{\alpha}{2}\right) - \psi\left(\frac{\alpha-1}{2}\right)\right) + \alpha \sum_{k\geq 1} \frac{(-1)^k}{k+\alpha}\left(\zeta(k+1)-1\right) \tag{7}$$

where ψ is the digamma function (the proof is ommitted here).

This formula is useful from a computational view point since the series converges geometrically fast. Thus it is possible to perform many checks of the main conjecture using functions f different than $f(x) = x$ and to make comparisons.

Regarding 2.

Althoug it is difficult to support experimentally the main conjecture using Riemann sums we provide some examples in APPENDIX 1 supporting somewhat the weak form of the main conjecture. To support the main conjecture for sums involving more terms in the summand we will introduce a conjectural quasi Monte Carlo method which apparently is working for function of unbounded variation like the fractional part function.

3.1 Low discrepancy sequences

It is a celebrated result that for any equidistibuted sequence $(a_n)_{n\in\mathbb{N}}$ in the interval $[0,1]$ we have for any Riemann integrable function h

$$I_h = \int_0^1 h(t)dt = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(a_k)$$

moreover if the sequence $(a_n)_{n\in\mathbb{N}}$ is a low discrepancy sequence and f is of bounded variation we have the Koksma–Hlawka inequality in dimension 1

$$\left| I_h - \frac{1}{n}\sum_{k=1}^{n} h(a_k) \right| \leq V(f) D_n^\star\left(a_1, a_2, ..., a_n\right)$$

where $V(f) > 0$ depends only on f and $D_n^\star(a_1, a_2, ..., a_n)$ denotes the star discrepancy of the n values $\{a_i\}_{1\le i\le n}$. For known computable sequences $(a_n)_{n\in\mathbb{N}}$ [2]it can be shown that

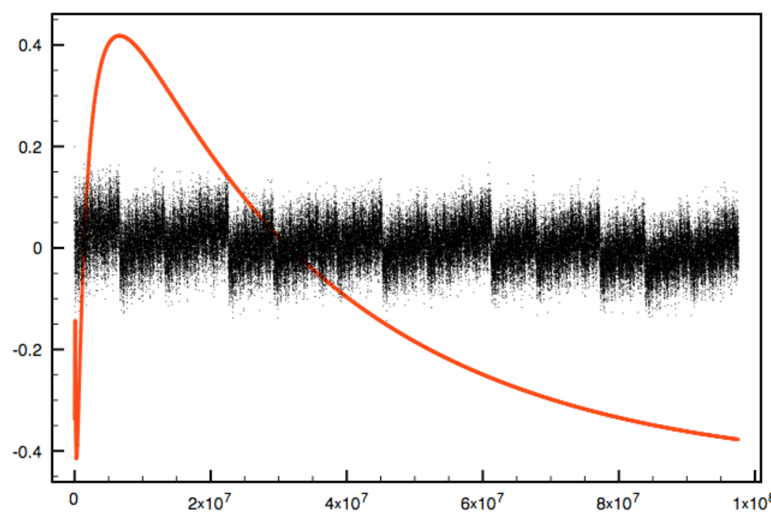
$$D_n^\star(a_1, a_2, ..., a_n) \ll \frac{\log n}{n}$$

Hence low discrepancy sequences $(a_n)_{n\in\mathbb{N}}$ provide a useful numerical method for approaching I_h compared to Riemann sums which require the computation of n distinct new terms at each stage. This method is the so called quasi Monte Carlo method (QMCM for short). In fact we think we can extend somewhat the Koksma–Hlawka inequality to functions of unbounded variation. Let us start with a simple example of function f which is bounded and continuous in $]0, 1]$ but is of unbounded variation. Namely we consider $h(x) = \sin(\log x)$. It is an interesting example since the Riemann sum is easily computable due to the fact

$$\sum_{k=1}^{N} h\left(\frac{k}{N}\right) = \cos(\log N)\sum_{k=1}^{N}\sin(\log k) - \sin(\log N)\sum_{k=1}^{N}\cos(\log k)$$

and we have $I_h = \frac{1}{2}$. So that we can compare the Riemann sum to the QMCM using some low discrepancy sequences. We know that $(\{rn\})_{n\in\mathbb{N}}$ is a low discrepancy sequence when $r > 0$ is any irrational value. Since this sequence is very easy to generate we will use in the sequel the sequence $\left(a_n = \left\{\sqrt{2}n\right\}\right)_{n\in\mathbb{N}}$ for our experiments. The following graphic compares the behaviour of $\sum_{k=1}^{N} h\left(\frac{k}{N}\right) - \frac{N}{2}$ and $\sum_{k=1}^{N} h\left(\left\{\sqrt{2}k\right\}\right) - \frac{N}{2}$.

Plot of $\frac{\sum_{k=1}^{N}\sin\log\left(\left\{\sqrt{2}k\right\}\right) - \frac{N}{2}}{\log(N+1)}$ (black) vs $\sum_{k=1}^{N}\sin\log\left(\frac{k}{N}\right) - \frac{N}{2}$ (red)



[2]For instance the van der Corput sequence, Halton sequences, Sobol sequences.

It is clear both graphics are bounded, which led us to the following conjecture.

3.2 QMCM for functions of unbounded variation

We state a conjecture for continuous functions then for discontinuous functions related to a possible QMCM

3.2.1 Conjecture on a QMCM for continuous functions of unbounded variation

Suppose h is a Riemann integrable function on $[0,1]$ which is bounded and continuous on $]0,1]$ but which is of unbounded variation on $]0,1]$. Suppose $(a_n)_{n\in\mathbb{N}}$ is a low discrepancy sequence on $[0,1]$. Then we have

$$\sum_{k=1}^{N} h\left(a_k\right) = I_h N + O(\log N) \Rightarrow \sum_{k=1}^{N} h\left(\frac{k}{N}\right) = I_h N + O(1)$$

3.2.2 Conjecture for discontinuous functions of unbounded variation

Suppose h is a Riemann integrable function on $[0,1]$ which is bounded and discontinuous on $]0,1]$ with countably many points of discontinuity and which is of unbounded variation on $]0,1]$. Suppose $(a_n)_{n\in\mathbb{N}}$ is a low discrepancy sequence on $[0,1]$. Then we have

$$\sum_{k=1}^{N} h\left(a_k\right) = I_h N + O(g(n)) \Rightarrow \sum_{k=1}^{N} h\left(\frac{k}{N}\right) = I_h N + O(g(n))$$

3.3 Experiments supporting the main conjecture 1.1.

The conjecture 3.2.2. is perhaps too general hence we make here a weaker conjecture related to the fractional part function only and involving sequences $(\{rn\})_{n\in\mathbb{N}}$ ($r>0$ irrational) as low discrepancy sequences.

3.3.1 Conjecture related to the fractional part

Suppose f is a continuous function of bounded variation on $[0,1]$. Let us define h as $h(x) = f\left(\left\{\frac{1}{x}\right\}\right)$. Then we have for any irrational value $r>0$

$$\sum_{k=1}^{N} h\left(\{rk\}\right) = I_h N + O(g(n)) \Rightarrow \sum_{k=1}^{N} h\left(\frac{k}{N}\right) = I_h N + O(g(n))$$

However the converse doesn't hold even using slowly varying function. This conjecture could help to improve experimentally the upper bound in both DDP and GCP. But what is interesting to us is the following last conjecture on the subject.

3.3.2 The comparison conjecture

The above conjecture led us to suppose that if 2 sums $\sum_{k=1}^{N} h_1(\{rk\})$ and $\sum_{k=1}^{N} h_2(\{rk\})$ behave similarly for 2 distinct functions h_1 and h_2 of form $h_1(x) = f_1\left(\left\{\frac{1}{x}\right\}\right)$ and $h_2(x) = f_2\left(\left\{\frac{1}{x}\right\}\right)$ so do the Riemann sums $\sum_{k=1}^{N} h_1\left(\frac{k}{N}\right)$ and $\sum_{k=1}^{N} h_2\left(\frac{k}{N}\right)$. More precisely we make this comparison conjecture.

Comparison conjecture

Let $h_1(x) = f_1\left(\left\{\frac{1}{x}\right\}\right)$ and $h_2(x) = f_2\left(\left\{\frac{1}{x}\right\}\right)$ given 2 continuous functions f_1, f_2 of bounded variation on $[0,1]$ and let $r > 0$ be an irrational value. Suppose there is a sharpest function u such that we have

- $\sum_{k=1}^{N} h_1(\{rk\}) = I_{h_1}N + O(u(n))$ and $\sum_{k=1}^{N} h_2(\{rk\}) = I_{h_2}N + O(u(n))$

Then there is a sharpest function v such that we have

- $\sum_{k=1}^{N} h_1\left(\frac{k}{N}\right) = I_{h_1}N + O(v(n))$ and $\sum_{k=1}^{N} h_2\left(\frac{k}{N}\right) = I_{h_2}N + O(v(n))$

In other words we can perform computation using the simple low discrepancy sequence $(\{rn\})_{n\in\mathbb{N}}$ and then deduce the Riemann sums behave similarly. This way it is now possible to provide experimental support for the main conjecture 1.1. and for larger values of n than in the APPENDIX 1.

So in the APPENDIX 2 we provide various comparisons of

- $\sum_{k=1}^{N} h_1(\{rk\})$ vs $\sum_{k=1}^{N} h_2(\{rk\})$

letting

- $h_1(x) = \left\{\frac{1}{x}\right\}$ and $h_2(x) = \left\{\frac{1}{x}\right\}^{\alpha}$ thanks to the formula (5).

4 Generalisations

4.1 The Riemann index of a function

The fact the fractional part function has a general intrinsic property is reminiscent of our tauberian approach to RH [Clo, Clo2] where we define the index of functions of good variation and where we conjecture that the function $x \mapsto \frac{\lfloor x \rfloor}{x}$ is a function of good variation of index $\frac{1}{2}$. By analogy we define the Riemann index of a function f, Riemann integrable on $[0,1]$, as follows.

Let

$$I_f = \int_0^1 f(t)dt$$

$$R_f(N) = \sum_{k=1}^{N} f\left(\frac{k}{n}\right) - I_f N$$

Then θ_f is the Riemann index of the function f if we have:

$$\theta > \theta_f \Rightarrow \lim_{N\to\infty} \frac{R_f(N)}{N^\theta} = 0$$

$$\theta \leq \theta_f \Rightarrow \lim_{N\to\infty} \frac{R_f(N)}{N^\theta} \neq 0$$

So that for continuous function f of bounded variation the index is trivially zero and for the non trivial function $f(x) = \left\{\frac{1}{x}\right\}$ it would be $\frac{1}{4}$.

Remark As for the main conjecture 1.1. we conjecture the index θ_f is also working for all remainders of type

$$R_f(N, \lambda, \mu) = \lambda \sum_{1\leq k\leq N/\lambda} f\left(\frac{k}{\lambda n + \mu}\right) - I_f N$$

where $\lambda \geq 1$ and $\mu \geq 0$ are real values, i.e.

$$\theta > \theta_f \Rightarrow \lim_{N\to\infty} \frac{R_f(N, \lambda, \mu)}{N^\theta} = 0$$

$$\theta \leq \theta_f \Rightarrow \lim_{N\to\infty} \frac{R_f(N, \lambda, \mu)}{N^\theta} \neq 0$$

4.2 Higher order

We could extend the previous ideas to d_m the divisors functions of order $m \geq 1$ given by the Dirichlet series

$$\zeta(s)^m = \sum_{n\geq 1} \frac{d_m(n)}{n^s}$$

To do that we would have to consider the $(m-1)$-dimensional integral.

$$I_{m-1} = \int_{[0,1]^{m-1}} \left\{\frac{1}{x_1 x_2 ... x_{m-1}}\right\} dx_1 dx_2 ... dx_{m-1}$$

and corresponding $(m-1)$-dimensionnal Riemann sums. Our belief is that there is no change and the Riemann index is the same whatever the dimension you consider. Hence we could extend the main conjecture in 1.1. as follows.

Suppose $r_{m-1}(N)$ is the number of terms in any corresponding $(m-1)$-dimensional Riemann sum for I_{m-1}, say $S_{m-1}(N)$, then we would have

$$S_{m-1}(N) = I_{m-1} r_{m-1}(N) + O\left(r_{m-1}(N)^{\frac{1}{4}+\varepsilon}\right)$$

Yielding

$$\sum_{k=1}^{n} d_m(k) = nP_{m-1}(\log n) + O\left(n^{\frac{1}{4}+\varepsilon}\right)$$

where P_{m-1} is a polynomial of degree $m-1$ with real coefficients. For $d=3$ we then would have

$$\sum_{k=1}^{n} d_3(k) = n\left(\log(n)^2 + (3\gamma - 1)\log n + (3\gamma^2 - 3\gamma - \frac{3}{2}\gamma_1 + 1)\right) + O\left(n^{\frac{1}{4}+\varepsilon}\right)$$

where γ_1 is the first Stieltjes constant. In fact we conjecture the more precise estimate

$$\sum_{k=1}^{n} d_m(k) = nP_{m-1}(\log n) + O\left(n^{\frac{1}{4}}\log(n)^{m-1}\right)$$

Like for QMCM in high dimension the factor $\log(n)^{m-1}$ is a constraint for numerical investigations in order to check the validity of this conjecture.

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APPENDIX 1

In order to check the weak form of the conjecture 1.1. we consider the remainder

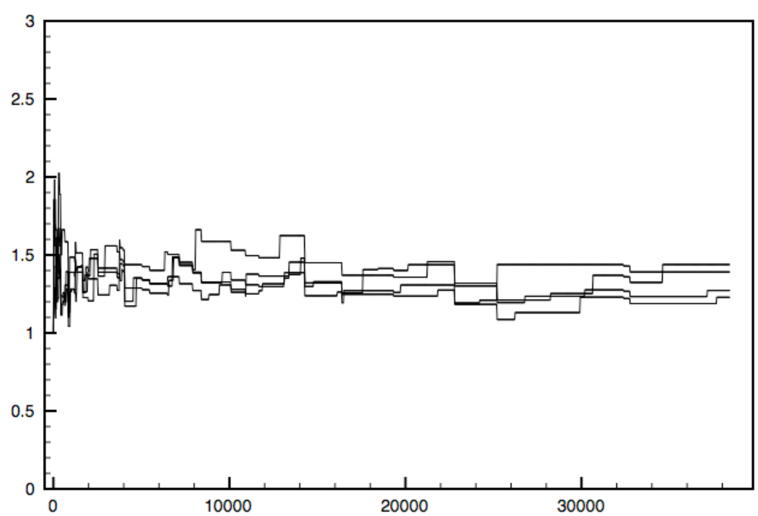
$$\Delta(N,\lambda,\mu) := \lambda \sum_{1\le k\le \frac{N}{\lambda}} \left\{ \frac{N}{\lambda k+\mu} \right\} - (1-\gamma)\, N$$

and we define

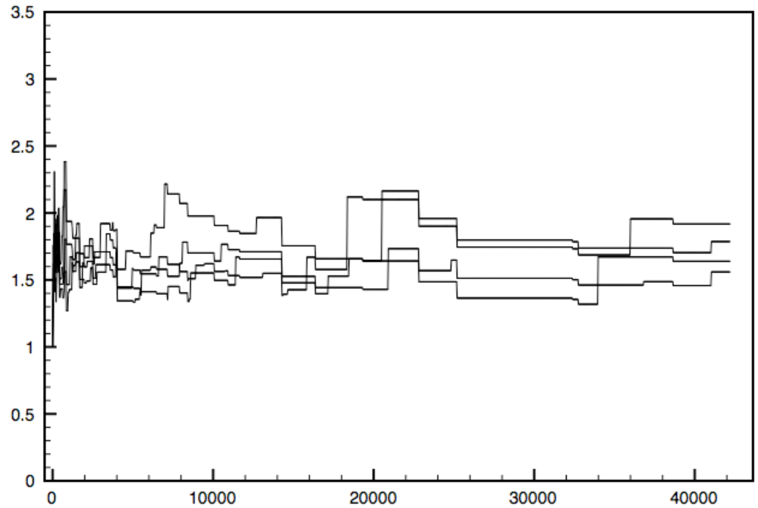
$$M(N,\lambda,\mu) = \max\left\{ |\Delta(k,\lambda,\mu)| \ : \ 1 \le k \le N \right\}$$

Next we compare $M(N,\lambda,\mu)$ to $M(N,1,0)$ for various (λ,μ) by plotting the ratio $\frac{M(N,\lambda,\mu)}{M(N,1,0)}$. We claim these ratio are always bounded supporting somewhat the weak of the conjecture 1.1. and assuming $M(N,1,0) = O(N^{1/4+\varepsilon})$.

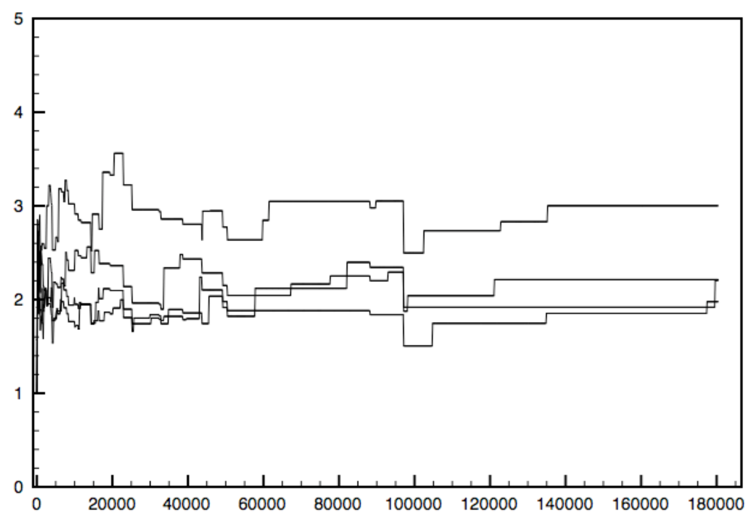
$\frac{M(N,\lambda,\mu)}{M(N,1,0)}$ for $(\lambda,\mu) = (5,1)\,,(5,2)\,,(5,3)\,,(5,4)$ and for $N \le 40000$



$\frac{M(N,\lambda,\mu)}{M(N,1,0)}$ for $(\lambda,\mu) = (7,1), (7,2), (7,3), (7,4)$ and for $N \leq 50000$



$\frac{M(N,\lambda,\mu)}{M(N,1,0)}$ for $(\lambda,\mu) = (10,1), (10,3), (10,5), (10,7)$ and for $N \leq 180000$



Although it seems clear these graphics are bounded we must be cautious since we computed few terms.

APPENDIX 2

Here we will use the conjectural trick in 3.3.2 to compare Rieman sums for

- $I_1 = \int_0^1 \left\{\frac{1}{t}\right\} dt$

and for

- $I_\alpha = \int_0^1 \left\{\frac{1}{t}\right\}^\alpha dt$

using the formula (5) in section 3. So let us define

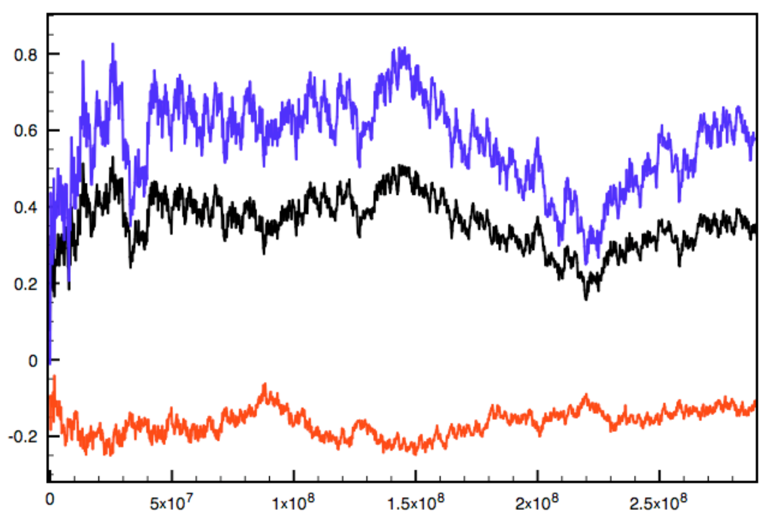
$$\Delta(N,\alpha) = \sum_{k=1}^{N} \left\{ \frac{1}{\{k\sqrt{2}\}} \right\}^\alpha - I_\alpha N$$

and

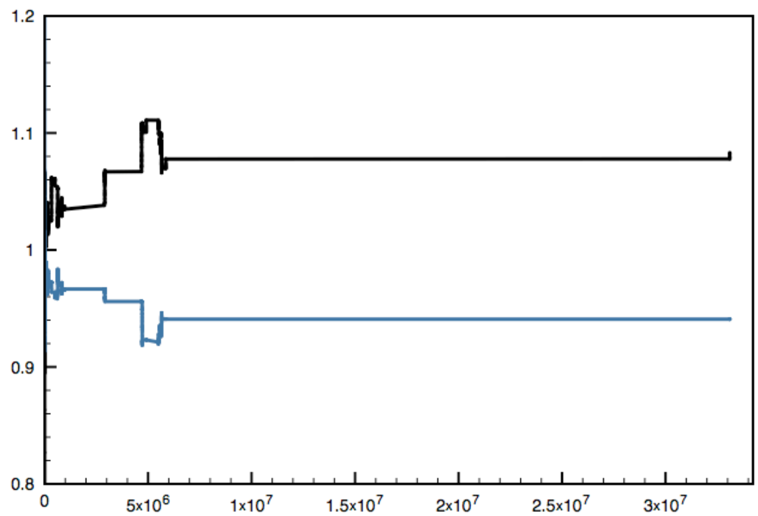
$$M(N,\alpha) = \max\left\{ |\Delta(k,\alpha)| \ : \ 1 \le k \le N \right\}$$

We plot $\frac{M(N,\alpha)}{M(N,1)}$ for various values of α and confirm somewhat that $\Delta(N,\alpha)$ behaves like $\Delta(N,1)$. Consequently using the conjecture 3.3.2 we support the stronger form of the main conjecture 1.1. But before this let us see how for distinct values of α the sums $\sum_{k=1}^{N} \left\{ \frac{1}{\{k\sqrt{2}\}} \right\}^\alpha$ behave similarly.

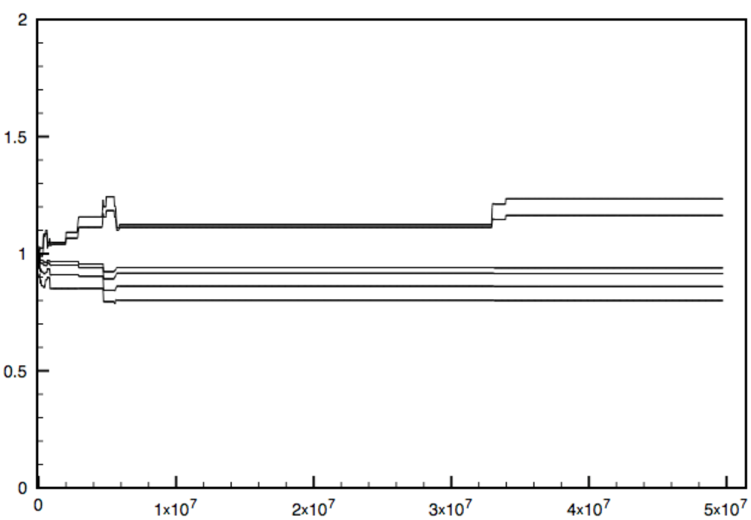
$N^{-1/4} \sum_{k=1}^{N} \left\{ \frac{1}{\{k\sqrt{2}\}} \right\}^\alpha$ for $\alpha = \frac{1}{2}$ (black), $\alpha = \frac{3}{2}$ (orange),$\alpha = \frac{1}{3}$ (blue).



$\frac{M(N,\alpha)}{M(N,1)}$ for $\alpha = \frac{2}{3}$ (black) and $\alpha = \frac{4}{3}$ (blue)



$\frac{M(N,\alpha)}{M(N,1)}$ for $\alpha = 2, 3, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}$.



We think above graphics are bounded or bounded by a slowly varying function.