A continued fraction expansion for the constant log(16)

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We start with the identity [Borwein et al., p. 495]

log(1 - z) + z = - (1/2)*z^2 * hypergeom([1, 2], [3], z) ...(1)

Applying Pfaff's transformation [Wikipedia]

hypergeom([a, b], [c], z)

= 1/(1 - z)^a * hypergeom([a, c - b], [c], z/(z - 1))

to (1) gives

log(1 - z) + z = - (1/2)*z^2/(1 - z) * hypergeom([1, 1], [3], z/(z - 1)).

Setting z = 1/2 gives

log(2) - 1/2 = (1/4) * hypergeom([1, 1], [3], -1).

Hence, by Gauss's continued fraction [Wikipedia or Borwein et al. Equation 8],

log(16) = 2 + 1/(1 + (1*2)/(2*3)/(1 + (1*2)/(3*4)/(1 + (2*3)/(4*5)/(1 + (2*3)/(5*6)/(1 + ...))))).

By means of equivalence transformations this can be put in the form

log(16) = 2 + 1/(1 + 1/(3 + (1*2)/(4 + (2*3)/(5 + (2*3)/(6 + (3*4)/(7 + (3*4)/(8 + ...))))))).

References

Jonathan Michael Borwein, Kwok-Kwong Stephen Choi, and Wilfried Pigulla, Continued Fractions of Tails of Hypergeometric Series, *Amer. Math, Monthly*, Vol. 112, No. 6 (Jun. - Jul., 2005), pp. 493-501

Wikipedia, Hypergeometric Function