Integer Sequences K-12

Euler and the Strong Law of Prime Numbers

AMM 95 (1988) 3-29

MM 63 (1990) 3-20

K feedback: Hours Struck by the Clock

Teachers: should we use 24-hour clock or am/pm?

lines connecting, 5, 12

kinesthetic (idea about kids jumping around circle

manipulative (circular geoboard using rubber bands)

Students:

Curricular:

Kindergarten

Hours Struck By A Clock!

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, 4, 5...

K: What’s the hook?

- Build on prior understanding of time and schedules. Use student’s own experiences to introduce ideas!
- What do you do at 8:00am? What time do you eat lunch? What time do you go to sleep?

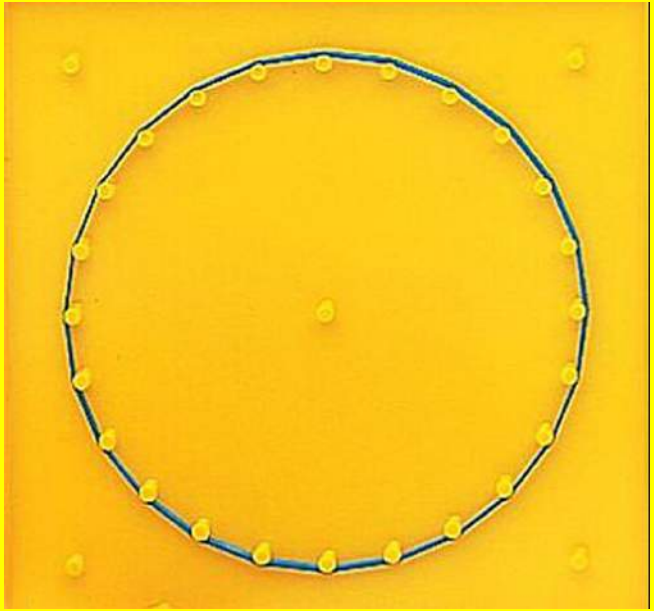
K: Why teach clocks?

- Repeating patterns are a foundational idea
- Leads to deeper understanding in divisibility and patterns
- Can teach many curricular outcomes using a clock - counting on, mental math, symmetry, patterns, etc.

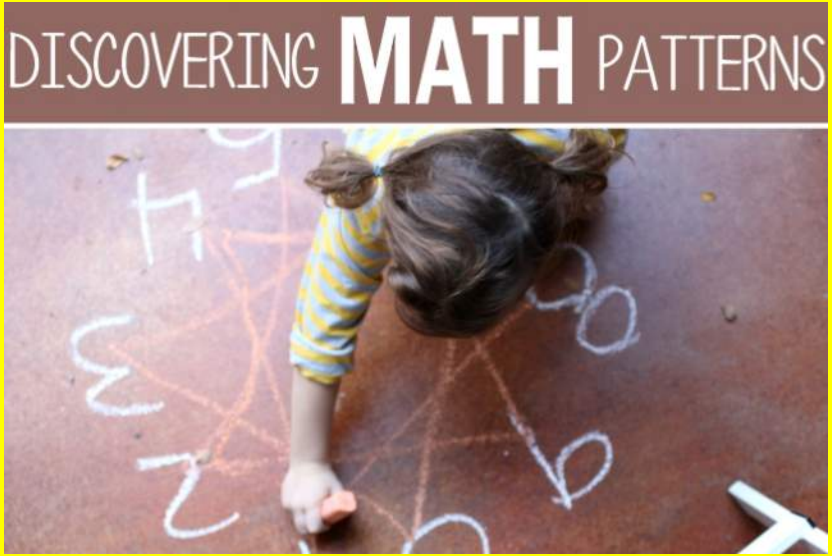
K: What do teachers need to know?

- Exploration of different patterns on a clock
  - If you count by 2’s, you’ll always land on the same numbers on every round through the clock
  - If you count by 5’s, you’ll land on different numbers sometimes but the pattern will still repeat
- Counting on “through” 12 - If you got to school at 9:00am and went home at 3:00pm, how many hours did you stay at school?
- Emphasize the hour numbers on the clock, NOT

K: Math Manipulatives for Clock



Circle Geoboards - also available as an app



Use chalk to make circle and have children explore patterns with the clock numbers

Grade 1:Tri-animals feedback

Teachers:

disadvantages: lack of triangular graph paper, difficult to draw equilateral triangles on board

counter-argument: teachers can create triangular paper with dots or lines online

using triangles hints at the idea that the unit of area is arbitrary

new medium for doodling

triangular paper commonly used in engineering

(Equilateral triangular graph paper can be printed from http://incompetech.com/graphpaper/trianglehex.html for free, for example.)

Students:

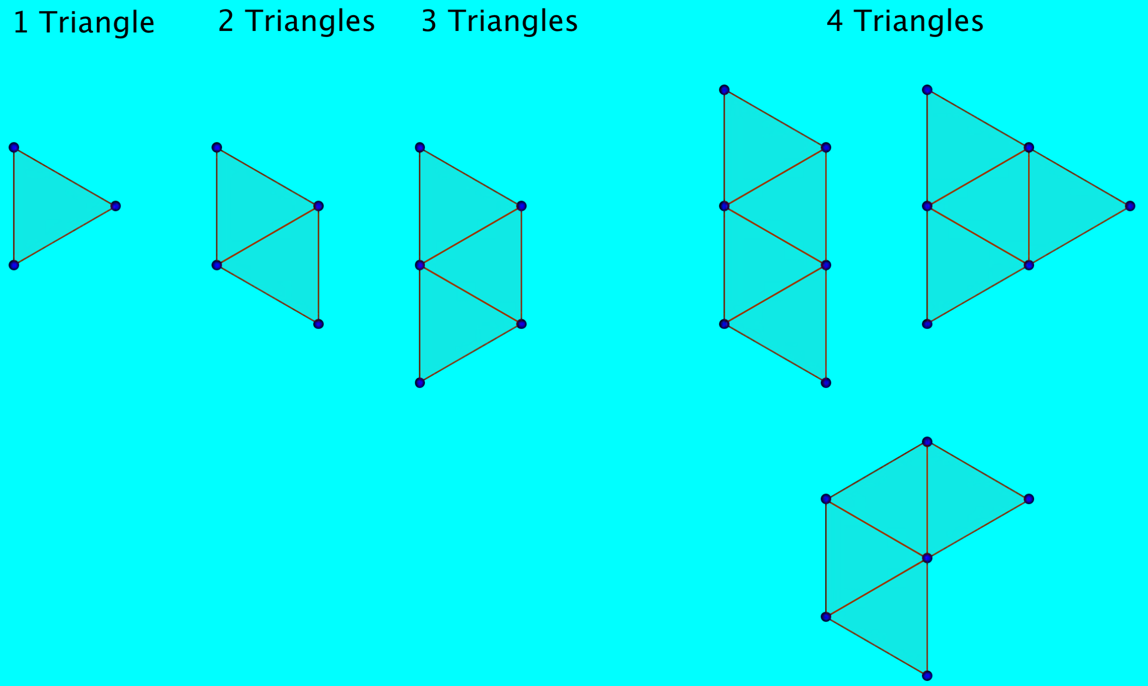
Curricular:

1st-2nd grade: curricular standard is measuring with non-standard unit of measurement

1

Tri-animals!

1, 1, 1, 3, 4,



1: What’s the hook?

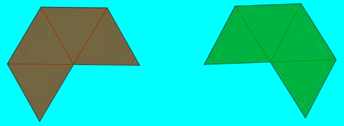
- Use manipulatives (equilateral triangles) to build a few tri-animals.
- Ask students to describe what they notice, patterns, etc.
- Allow students time to be creative and build their own.

1: Why teach tri-animals?

- Research has shown that spatial reasoning is fundamental at an early age for success in future math and language
- Focus on manipulating figures, not on area or perimeter
- Classification or sorting problem

1: What do teachers need to know?

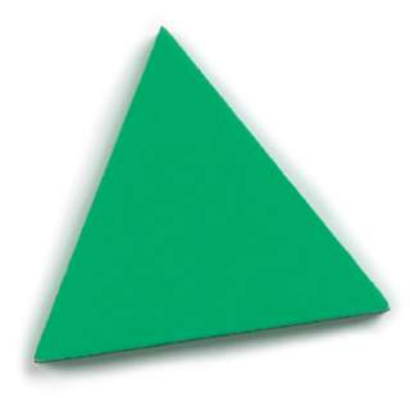
- Help students to count systematically to realize when all possibilities have been counted
- Need to think about whether reflections or rotations of tri-animals are allowed
  and include students' debate
- Infuse language of rotation, reflection

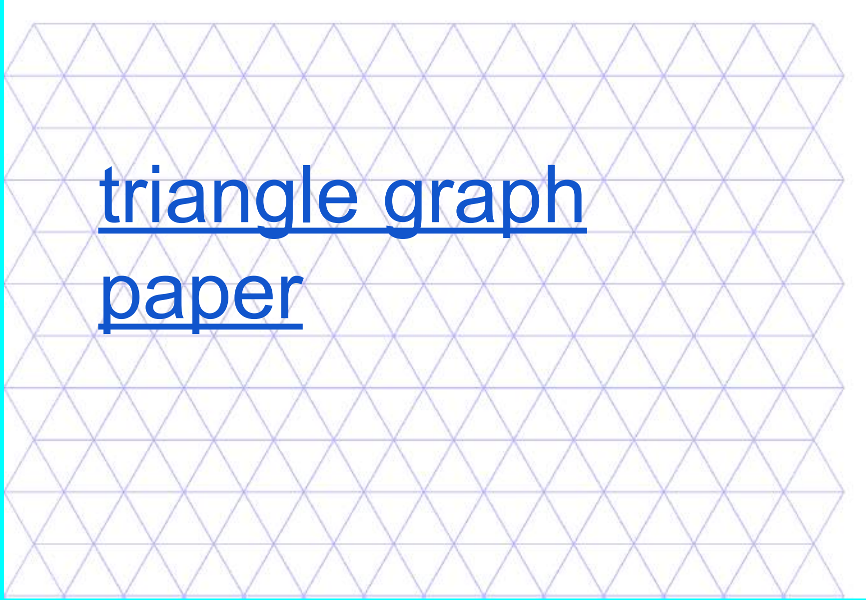


1st: Math Manipulatives for Tri-Animals

Recommended: first use hands-on manipulatives, then paper

Use equilateral triangles - found in pattern block sets. Also, students can glue together triangles made with Ellison block cutter

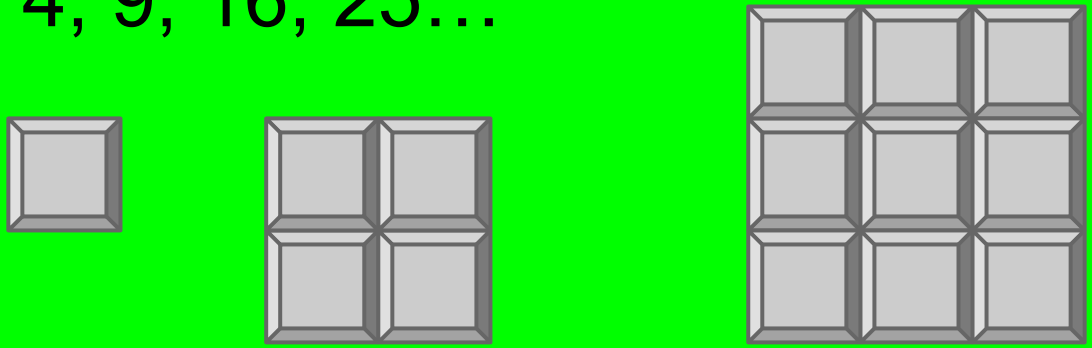




2

Square Numbers!

1, 4, 9, 16, 25…



2: What’s the hook?

- Use manipulatives such as color tiles to build first few squares
- Ask students to describe what they notice, patterns, etc.
- Connect to prior understandings of shapes and properties of shapes

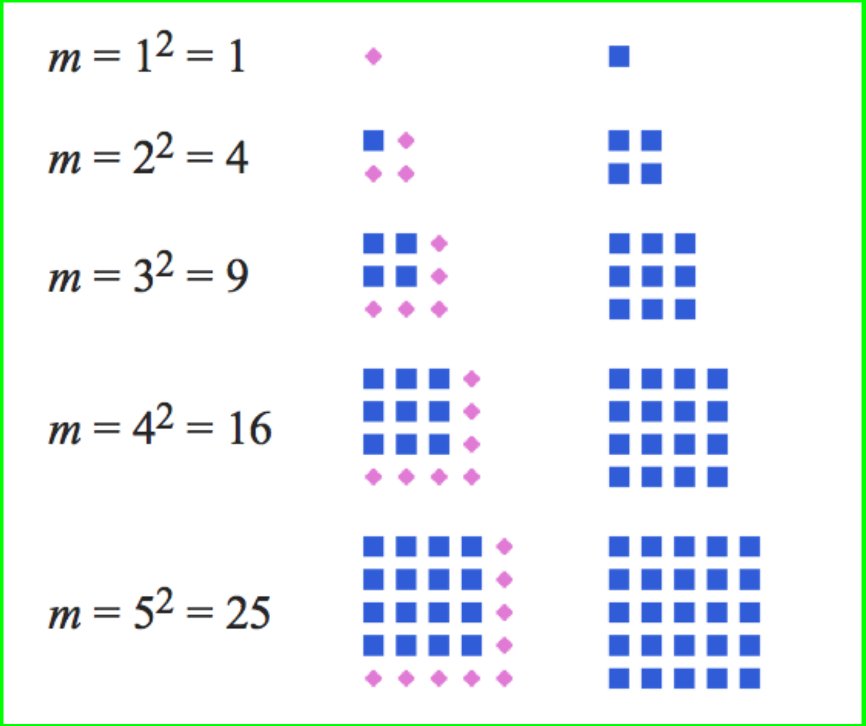
2: Why teach square numbers?

- Students need to make the connection that a square number can be represented as a square shape!
- Lines of symmetry, properties of a square remain the same regardless of size
- Lots of relationships to odd and figurate numbers
- Visual proof of why consecutive odd numbers starting at 1 always sum to a square number (see next slide)
- Possible intro to multiplication using the array or area model

2: What do teachers need to know?

- The proof of why consecutive odd numbers add to a square number (on next page)
- Strategies for mental math
- Predict what the next square number will look like
- Some knowledge of triangular numbers would be helpful

2: Why odd numbers sum to a square

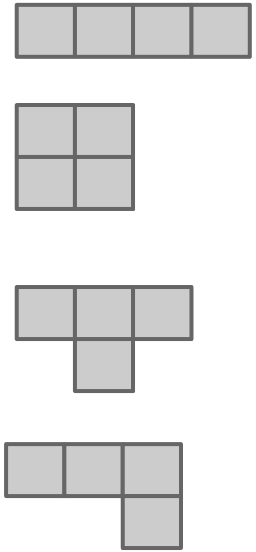


2: Think about...

- Having color tiles, counters, and/or graph paper available at all times for students to use
- Reinforce student understanding of squares and the properties of a square
- How square numbers relate to triangular numbers (1, 3, 6, 10, 15, etc.) http://www.themathpage.com/arith/appendix.htm
-

Grade 3: Polyanimals!

The students use n 1x1 unit squares to work out the minimum and maximum perimeter. All of these use 4 tiles but three have P=10 and one has P=8. Extension: Group these shapes (animals) by height considering that the animals lie down on their longest side. Can have students work with fewer or more squares/explore other shaped tiles.



Another sequence to explore is the number of different polyanimals with n tiles.

Teacher needs to know: the definitions of area and perimeter. Be ready to discuss reflections and rotations--they look different but can be considered the same. Model how to count sides in a systematic way with ‘tally marks’ on each edge. There are five sequences mentioned:

1. Minimum perimeter given a fixed area
2. Maximum perimeter given a fixed area
3. Area with a fixed perimeter
4. Height with a fixed area or permimeter
5. Are reflections and rotations included?

Curricular links: Describe the characteristics of 2-D shapes,counting, repeating patterns, organisation, communication, demonstrate an understanding of perimeter of regular and irregular shapes.

Grade 3: Polyanimals feedback

Teachers: Have examples of what the sequence would look like (i.e., 5 squares), need organizing system like height because 6 square sequence has many possibilities

Students: Some students will want to go further with the sequence, but need help organizing

Can organize with height, Perimeter, Area, Children (one can fit in another), symmetry.

Curricular Links:

Grade 4: Square Pairs

Henri Picciotto proposed this sequence.

Using all of the counting numbers from 1 to 2n, pair them up so that each pair adds up to a perfect square, i.e., the set of numbers 1 to 8 has four ‘square pairs’ (n=4) 1+8=9, 2+7=9, 3+6=9, 4+5=9. More generally, what are the positive integers 2n such that the numbers 1 to 2n can be organised in n pairs whose sum is a perfect square?

For teachers: Explore making different rectangles w/ a given number of tiles. Relationship between squares and rectangles. Now consider square numbers only: Students explore making squares with a varied number of tiles. Students now approach problem. (Scaffolding for those who need it: Using 4 tiles students work out pairs that add to 4. Students take enough tiles to make the next square (9) and then, using tiles, work out the pairs that add up to a square number.)

Curricular links: Communication (working with a partner, explaining their process using mathematical language); Mental mathematics (fluency); Number sense (addition, multiplication), Patterns (recognize pairs that make 2n); general problem solving skills, leads to a deeper understanding of square numbers. Identifying primes and composites. Teachers should note that the square-pairs numbers (n) are 4, 7, 8, 9 and all numbers greater than 11. This problem allows students to make the surprising discovery that any number above 11 will work as n.

Grade 4: Square Pairs feedback

Teachers:

Students:

Curricular Links:

this sequence will be hard sell because no obvious curricular link?

perhaps more 3rd grade level - practicing simple multiplication and addition

counter-argument: fluency practice

do it with primes? (US standard: distinguish prime from composite from 1-100)

Can make a chain or a necklace where every adjacent pair sums to a square (but requires 1-30 at least to get going)

Grade 5?: Recaman div, subt, add, mult

This sequence was created by a 15 year student.

A254873

Grade 7?: Funky-Fractions-To-1 Sequence.

David Wilson suggested this question. It is not currently in the OEIS.   4,9,7,20,6,33,13,23,16,62?,8,75?,18,17,25,...

Starting with 1, on the first step add 1/n, and on subsequent steps either add 1/n or take the reciprocal. What is the smallest number of steps needed to return to 1? This number of steps is the nth term of the sequence. (Note: the n=0 and n=1 terms are not defined, so the sequence actually starts with the 2nd term.)

Grade 7? (Example)

Consider n=3. At each step after the first one, we can either add 1/3 or take the reciprocal.

1-->4/3-->5/3-->2-->7/3-->8/3-->3-->⅓-->⅔-->1

OR

1-->4/3-->5/3-->2-->½-->⅚-->7/6-->3/2-->⅔-->1

Both sequences contain 9 steps. A brute force search reveals that this number of steps is optimal. Therefore, the n=3 term of the Funky-Fractions-To-1 sequence is 9.

Grade 7 (Curricular Links)

This activity involves adding fractions with like and unlike denominators, working with improper fractions, as well as problem solving and organizational skills. Patterns relating to number factorization are also involved.

Grade 7: Funky-Fractions-To-1 feedback

Teachers:

Possible Task:

Leaderboard (advertise what the best one is)...do it in little chunks...repeated approaches

Students: likely will not find the optimal, but their best fraction

Curricular Links:

Grade 6: Ron Graham's Sequence

OEIS 6255

For a given number *n*, make a list of increasing numbers beginning with *n* so that the product of all the numbers in the list is a perfect square. Look for the list that ends with the smallest possible number. This is the n^{th} entry in Ron Graham's sequence.

Grade 6 (continued)

Example: Let n be 2. We could make the list 2,8. The product is a square number.

Alternatively, the list could be 2,3,6. The product of these numbers is also square, but the last number is smaller than 8. We can convince ourselves that 6 is the smallest ending number possible, so the 2nd entry in Ron Graham’s sequence is 6.

Grade 6 (continued)

Curricular Links:

Students in grade 6 should understand that natural numbers factor uniquely. This fact is used repeatedly while hunting for terms in Ron Graham’s sequence. The activity also lends itself to practice with exponential notation. Students also improve fluency with basic multiplication facts. Students also practice problem solving and organizational skills.

Grade 6: Graham’s Sequence feedback

Teachers:

Find a product that gives a square number

Try to make the last number to the smallest number

Needs a much more detailed explanation for the teachers.

Students: will likely reach optimal number

Curricular Links:

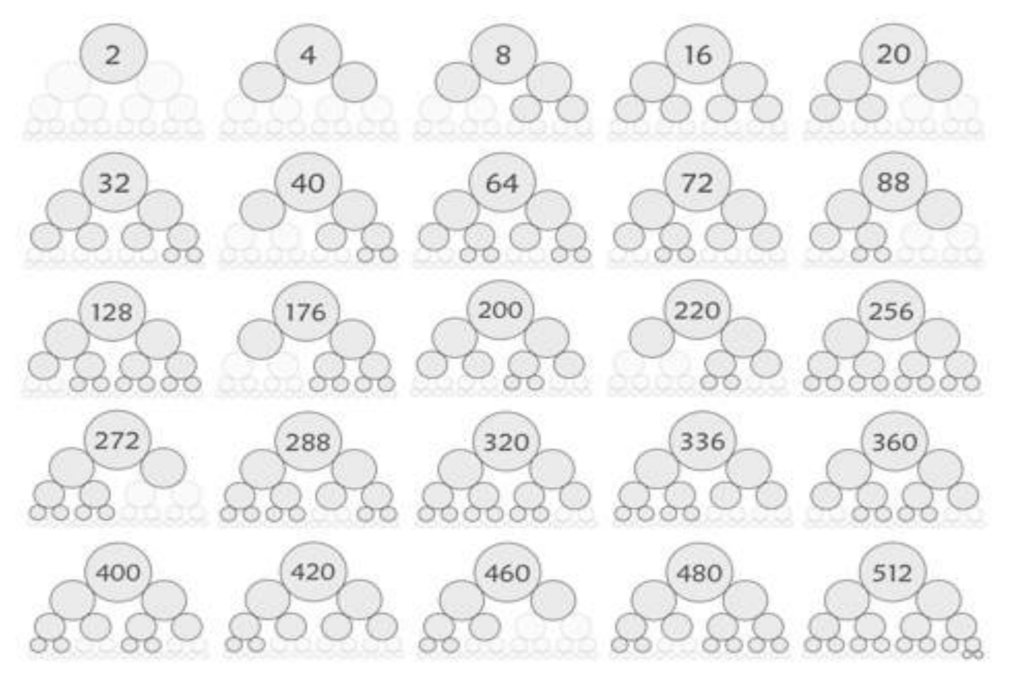
Grade 6? (A125508)

An integral fission factor tree (IFFT) is a factor tree with these two properties:

- children in each pair must be as close as possible
- the child on the left is less than or equal to the child on the right

A number is in the sequence if it is the first to create a new IFFT shape.

For example, 2 creates the first shape consisting of a single node. The shapes for 8 and 20 are different.



How to Introduce Integral Fission

1. *Do not explain the rules up front.* Draw an IFFT, with no numbers in it. 20 works well for this. Ask students for numbers to put in the top circle, and draw an IFFT for the numbers they suggest, explaining the rules as you go, until they suggest 20.
2. Now create an IFFT by implementing student suggestions for which circles should be split. Ask for the smallest number that would work in the top circle.
3. Ask students to find three consecutive integers between 2 and 50 that create the same IFFT. If a student pair solves - ask them to find three consecutive between 150 and 200.
4. Suggest that we should start with 2, and find every possible IFFT shape. If a shape has already been found, we don’t need it, and we move on. For example, 3, 5, 7, and 11 create trees that are the same as 2, so we do not include them on our list.

Curricular Connections

Integral fission provides an interesting set of challenges to practice prime factorization.

Grade 7: Integral Fission feedback

Teachers:

integral fission better than prime factor tree

counterargument: room for both the prime factor tree and integral fission

all primes result in 1st pattern

multiples in 2 result in 4 pattern

multiples in 3 result in 8 or 20 pattern

work with small numbers, give them a perspective shape first, find number that fits there

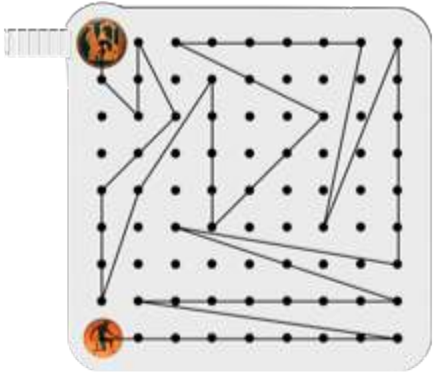
Students: should all generate the same tree

Curricular Links:

Grade 8: (A226595)

Lengths of maximal non-touching increasing paths in n X n grids starting at upper left - ending lower left.

Pythagoras



Grade 8: Increasing Path feedback

Teachers: your path must get longer as you move through the grid, seeking maximum number of steps; not adding a bunch of square roots

Students: find good solutions, but rarely the optimal solution; often find zig-zag pattern, but not optimal

do not often realize that you can minimize square distances

Curricular Links: this task can introduce Pythagoras

9 Staircase numbers

9 feedback

10: Szekeres's Sequence (A003278)

Construct a sequence beginning with 1, 2, … where you choose the smallest next integer that avoids a 3 term arithmetic progression. (at least 20 terms)

1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 82, 83,

Eg: 1, 2, 3, doesn't work because this forms a three arithmetic term progression. 1, 2, 4, 5, 6, doesn't work either.

Students can build it by identifying unsuitable terms.

Have different groups start with different first two terms and compare.

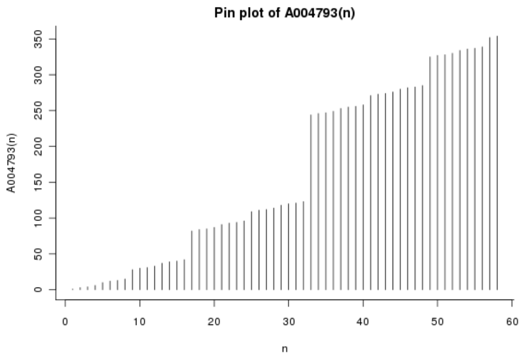
1, 3, 4, 6, 10, 12, 13, 15, 28, 30, 31, 33, …

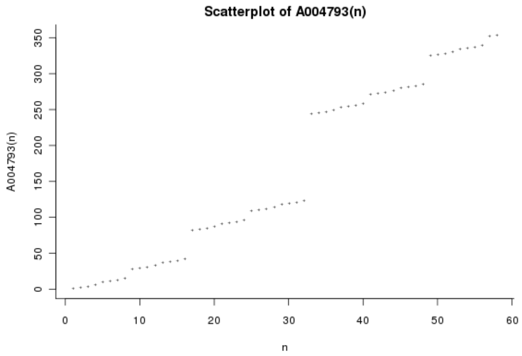
1, 4, 5, 8, 10, 13, 14, 17, 28, 31, 32, 35, …

1, 5, 6, 8, 12, 13, 17, 24, 27, 32, 34, 38, …

Curricular Connections: Introduce discrete, non-linearity, relations

Advanced: Students may detect ternary expansion to explain the numbers in the sequence.





10

- prev. knowledge: Pattern recognition
- misconceptions: expect linear behaviour, wrong notion of common difference
- expectations: students expect large jumps after small jumps. Students should work in groups
- challenges: phrasing of “arithmetic progression” - can employ “common differences” instead. Slower students can help a lot by checking the work done
- resources: graph paper and ruler
- class discussion: What patterns were discovered? How do the different sequences compare?
- WWYDT: Relations, linear vs non-linear, functions

Grade 10: Szekere’s Sequence feedback

Teachers: need to include another example in the intro slide

distance between successful terms...students will see patterns

if you shift the sequence, it does not change the property of arithmetic progression

can start with different terms, but similar structures (see graph)

simple problem if you know ternary expansion

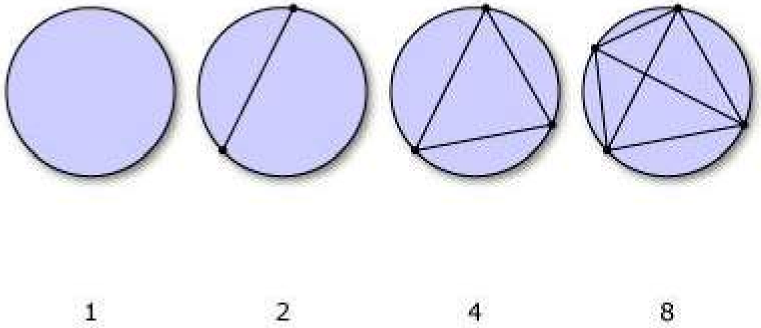
Students:

instead of starting with 1...prove that you can’t do any better if you start with 2

choose subset from 1 to n….in that case, you can’t do any better than

Curricular Links:

11



The Strong Law of Small Numbers

Have students create the sequence of numbers related to the maximal number of regions (no three lines intersect at a point) obtained by joining n points around a circle by straight lines (A000127):

1, 2, 4, 8, …

And compare this with the sequence of 2^n:

1, 2, 4, 8, …

Students may discover the prudence of not making careless conjectures based on a small number of examples.

Compare with Pascal’s Triangle

Many more examples are available in Richard Guy’s manuscript: http://www.ime.usp.br/~rbrito/docs/2322249.pdf

Curricular connection is the development of proofs where all examples work by using algebra.

11

- prev. knowledge: Detecting patterns
- misconceptions: Relying on a few examples to get an answer in general
- expectations: Students make incorrect conjecture, miscount at 31 (some will get 30 and 32), and attempt to fit the answer to the expected pattern, invariance of number of regions, some might not connect all their points
- challenges: Frustration with not getting the expected answer
- resources: knowledge of what a chord is, sheets with circles (some with dots on it)
- class discussion: Compare with powers of 2, Pascal’s triangle
- WWYDT: Be more cautious with conjectures, lead-in to proofs

Grade 11: Strong Law of Small Numbers feedback

Teachers: include info that students should be surprised by their findings and connection to Pascal's triangle in intro slide

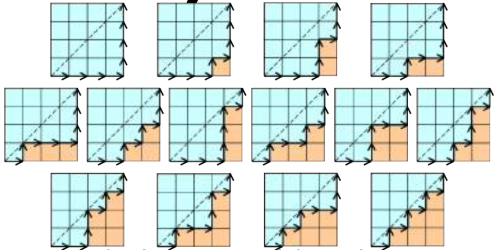
Goal is not the formula, but the process...similar to how mathematicians work -- formulating and testing conjectures; difference between thinking you have a pattern versus verifying the pattern

"Build up students and then crush them;)"

Students:

Curricular Links:

12: Catalan numbers (A000108)

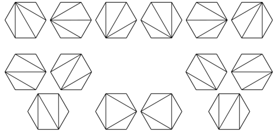


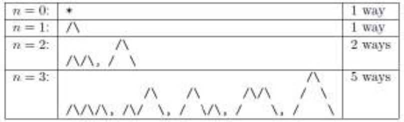
n = 4, there are 14 paths

The number of paths from bottom left to top right of an n x n grid which only move right and up, without crossing the diagonal. From n = 0, the sequence is: 1, 1, 2, 5, 14, 42, 132, 429,

Showcase a variety of equivalent problems and have students demonstrate the bijection between them. Examples include:

- The number of ways a convex polygon with n+2 sides can be cut into triangles
- The number of ways n pairs of parentheses can be correctly matched: for n = 3, ((())), ()(()), ()()(), (())(), (()())
- The number of ways to tile a stairspace shape of height n with 1 x n tiles
- The number of full binary trees with n+1 leaves
- The number of ways to form a “mountain ranges” with n upstrokes and n down-strokes that all stay above the original line.The mountain range interpretation is that the mountains will never go below the horizon.
- etc...





Mountain Ranges

Curricular Connection: Students can be shown how to get this sequence using combinations: (include difference equation 2nCn - 2nC(n+1))

$$t_n = \frac{1}{n+1}\,{}_{2n}C_n$$

12

One way to introduce this is to start with a “difficult” problem like the number of ways to write balanced pairs of parentheses, then develop bijection to “easy” problem like the first one, connecting it to previous knowledge. Then develop formula, etc

12

- prev. knowledge: functions, rational functions, combinations
- misconceptions: none
- expectations: expect working up to $n = 7$ or so by hand
- challenges:
- resources:
- class discussion: bijection to different problems
- WWYDT:

Grade 12: Catalan Numbers feedback

Teachers:

this problem appropriate for different bands of students (i.e., regular vs. AP/Honors)

how to present to teachers and students? otherwise they might get stuck

an easier entry point into the problem?

Students:

Curricular Links:

Racaman Sequence

found by student: $^{-}22$, $^{+}11$, $^{-}9$, $^{x}18$

More questions

- In what ways does the sequence build on students' previous knowledge, life experiences, and culture?
- What misconceptions will students likely have? What errors might they make?
- What are your expectations for students as they work on and complete the task?
- What particular challenges does the sequence present to struggling learners or ELL students?
- What resources will students have to use in their work, to give them entry into, and help them reason through, the task?
- How will you orchestrate a class discussion to accomplish your mathematical goals and make learning visible?
- What will you do tomorrow that will build on this lesson?

End of Conference Ideas

- Share doc about how we are sharing these integer sequences
- invite feedback from teacher, so we understand needs of the teachers in the field
- write articles for math teachers about these integer sequences (i.e., NCTM publications in US, Pi in the Sky,)