6/8/98

Two dimensional theta series of lattices (with E.M. R.)

From Ozeki - Washio, JRAM ....(96.1)
"Explicit formulae for the Fourier coefficient"...

$\mathcal{H}_n$ = Siegel upper half-space of degree n

= $n \times n$ complex symmetric matrices $Z = X + iY$

with $\operatorname{Im}(Z) = Y = \frac{1}{2i}(Z - \bar{Z}) > 0$

— I am now quoting from Kitaoka, "Lectures on Siegel modular forms" (Tata, 1986) ....(96.2)

Modular group (all this is from Kitaoka)

$$\Gamma_n = Sp(2n, \mathbb{Z}) =$$

$$= \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{Z}) : M J_n M^t = J_n \right\}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

acts on $\mathcal{H}_n$ as a discontinuous group of holomorphic automorphisms

$$Z \mapsto M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}$$

Note that $M\{Z\} = CZ + D$ is invertible for $M \in \Gamma_n$

Note: Γ_n is closed under transposition

Note: $M^{-1} = \begin{pmatrix} D^{tr} & -B^{tr} \\ -C^{tr} & A^{tr} \end{pmatrix}$.

Note: $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$

$$\Leftrightarrow \begin{cases} AD^{tr} - BC^{tr} = I_n \\ AB^{tr} = BA^{tr} \\ CD^{tr} = DC^{tr} \end{cases}$$

$\Gamma_{n,\infty}$ is the subgroup $\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n : (C = 0), \right.$

$A, D = A^{tr^{-1}} \in GL_n(\mathbb{Z})$,

$\left. S = A^{-1}B \text{ symmetric integral matrix} \right\}$

The next sentence from Kitaoka isn't clear:

If $M = \begin{bmatrix} * & * \\ C & D \end{bmatrix}$, and $N = \begin{bmatrix} * & * \\ C & D \end{bmatrix}$ are both in Γ

then $M = \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix} N \quad \Gamma_{n,\infty} N.$

For $Z \in \mathcal{H}_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,

$$\operatorname{Im} M\langle Z\rangle = (C\bar{Z} + D)^{tr^{-1}} \operatorname{Im}(Z)(CZ+D)^{-1} > 0.$$

Fund. domain $\Gamma \backslash \mathcal{H}_n$ is

$\mathcal{F}_n = \Big\{ Z \in \mathcal{H}_n :$ (1) $|\det(CZ+D)| \geq 1$ for every primitive integral $(C\ D)$ with $CD^{tr} = DC^{tr}$

(2) $\operatorname{Im}(Z)$ is M-reduced

(3) Elements of $X = \frac{1}{2}(Z + \bar{Z})$ are $\leq \frac{1}{2}$ in absolute value. $\Big\}$

Let
$$\mathcal{G}_n = \bigcup_{M \in \Gamma_{n,\infty}} M\langle \mathcal{F}_n \rangle \qquad (98.1)$$
$$= \bigcup_{\substack{U \in GL_n(\mathbb{Z}) \\ S = S^t \in M_n(\mathbb{Z})}} \left(\mathcal{F}_n[U] + S \right)$$
where $P[Q] = Q^{tr} P Q$

Then
$$Z \in \mathcal{G}_n \Rightarrow \min \operatorname{Im}(Z) \geq \frac{\sqrt{3}}{2} \qquad (98.2)$$

$q \in \mathbb{N}$ will be our ~~weight~~ "level"

$k \in \frac{1}{2}\mathbb{Z}$ will be our "weight"

$\Gamma_n(q)$ = principal congruence subgroup of level q in Γ_n
$$= \{ M \in \Gamma_n : M \equiv I_{2n} \ (\mathrm{mod}\ q) \}$$

etc etc etc

I now return to Ozeki–Washio (96.1) and other Ozeki papers.

From Klingen's book p. 100. The original source for Siegel mod forms is the following defn from quadratic forms:

Let A =

$d \times d$ m

Then

$\Theta(Z$

Prop (99.2

$\Theta($

– I'm not

of weig

multiplic

q

In parti

Let A = Gram mx $= \frac{1}{2}$-integral symm pos def
$d \times d$ matrix = Gram matrix of Λ in d dims.
Then

$$\Theta(Z,\Lambda) \triangleq \sum_{g \in \mathbb{Z}^{n\times d}} e^{2\pi i \, \mathrm{Tr}(Z[g])}, \quad Z \in \mathfrak{H}_n \qquad (99.1)$$

where $\quad Z[g] = g^{tr} Z g$

Prop (99.2) p. 100 Klingen

$\Theta(Z,\Lambda)$ is a modular form wrt

$$\Gamma_0(q) = \left\{ M \in \Gamma_n \;\middle|\; M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ C \equiv 0 \pmod{q} \right\}$$

— I'm not sure if it's the same Γ_n

of weight $\frac{d}{2}$ and with certain 8^{th} roots of 1 as multiplier system. Here

$$q = \text{level of quad. form } 2A$$
$$= \text{smallest } q \text{ s.t. } q(2A)^{-1} \text{ is integral}$$

In partic $\Theta(Z,\Lambda)$ is a modular form iff $q = 1$.

Continuing with Klingen pp 100–101:

We assume A to be half-integral.
It is more common to say that

$$B = 2A$$

is even and integral. Then we want (<u>if we are to get modular forms</u>) that B is an even unimodular lattice.

We get the expansion

$$\Theta(Z, A) = \sum_{s \geq 0} \alpha(s, A)\, e^{2\pi i \,\mathrm{Tr}(sZ)}$$

where

$$\alpha(s, A) = \#\{ g \mid s = g^{tr} A g, \ g \text{ integral} \}$$

= no of representations of the quad form s by A

Now assume $d < n$ (Klingen p. 101 middle)
– really ??

Anyway, let us consider

$$\Theta^{(2)}_{\Lambda}(Z) = \text{2-dimensional } \theta\text{-series of } \Lambda$$

$$= \sum_{T} \alpha(T,\Lambda)\, e^{2\pi i \,\mathrm{Tr}(TZ)} \qquad (101.1)$$

for $Z \in \mathcal{H}_2$, where $T \in \left(\tfrac{1}{2}\mathbb{Z}\right)^{2\times 2}$, ~~p.d.~~ p.s.d., $T \geq 0$

$$T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, b, c \in \tfrac{1}{2}\mathbb{Z},$$

and (*)

Then

$$\mathrm{Tr}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\underbrace{\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}}_{Z} = a\tau + 2bz + c\tau'$$

so

$$\Theta^{(2)}_{\Lambda}(Z) = \sum_{T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}} \alpha(T,\Lambda)\, e^{2\pi i (a\tau + 2bz + c\tau')} \qquad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \qquad (101.2)$$

<u>Theorem (101.3)</u> If Λ is any ~~integral~~ lattice then

$$\Theta_{A_2 \otimes \Lambda}(\tau) = \Theta^{(2)}_{\Lambda}\left(\begin{pmatrix} 2 & +1 \\ +1 & 2 \end{pmatrix}\tau\right) \qquad (101.4)$$

<u>Proof.</u> From (89.2), or from first principles, every vector of

---

(*) and $\alpha(T,\Lambda)$ = # pairs $u, v \in \Lambda$ such that

$u \cdot u = 2a$, $u \cdot v = b$, $v \cdot v = 2c$

~~i.e. $\begin{pmatrix} u \\ v \end{pmatrix}(u, v) = 2T$~~

$A_2 \otimes \Lambda$ corresponds to a triple

$$v, v', v'' \in \Lambda \times \Lambda \times \Lambda$$

with $v + v' + v'' = 0$

[For A_2 is gen'd by $(1,-1,0)$, $(0,1,-1)$,

so $A_2 \otimes \Lambda$ is gen'd by

$$(v,-v,0) \quad v \in \Lambda, \; \& \; (0,\omega,-\omega), \; \omega \in \Lambda$$

ie typical vector of $A_2 \otimes \Lambda$ is

$$\sum c_v (v,-v,0) + \sum d_\omega (0,\omega,-\omega)$$

$$= (v, v', v'') \text{ with } \underset{\text{in } A_2 \otimes \Lambda}{v + v' + v'' = 0} \quad]$$

The norm of this vector, described by

$$(v, v', -v-v'), \quad v, v' \in \Lambda,$$

is

$$2 v \cdot v + 2 v \cdot v' + 2 v' \cdot v'$$

So

$$\Theta_{A_2 \otimes \Lambda}(\tau) = \sum_{v, v' \in \Lambda} e^{4\pi i (v \cdot v + v \cdot v' + v' \cdot v')\tau}$$

OTOH

$$\Theta^{(2)}_{\Lambda}\Big(\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\tau}_{Z}\Big) = \sum_{T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}} \alpha(T, \Lambda) e^{2\pi i (2a\tau + 2b\tau + 2c\tau)}$$

$\tau = 2\tau$, $z = \tau$, $\tau' = 2\tau$ and these agree. $\square$

(103.1)
Fact If
then Θ
for Γ_1
— i.e. it is
The proof is
where?

Cor (103.2)
free ring

where E_i

Theorem (10

$A_2 \otimes$

shadow p

h is prob
pseudo-
resp., th

(103.1)

Fact If Λ is even unimodular lattice in $\mathbb{R}^d$, then $\Theta^{(2)}_\Lambda(Z)$ is a Siegel modular form of wt $\frac{d}{2}$ for Γ_1 w.r.t. trivial character.

— i.e. it is the space $M_2^{d/2}$ in Klingen's notation, p. 11

The proof is in Klingen, Ch IV, one hopes — but where?

Cor (103.2) (Igusa) $\Theta^{(2)}_\Lambda$ (if Λ is Type II) is in the free ring

$$\mathbb{C}\left[E_4, E_6, E_{10}, E_{12}\right] \qquad (103.3)$$

where E_i are Eisenstein series (Klingen p. 123)

Theorem (103.4) If Λ is even unimodular, then $A_2 \otimes \Lambda$ is 3-modular, and

$\Theta_{A_2\otimes\Lambda}(z)$ is in our ring — see the shadow paper, — ie is modular form for Γ_0^{3+}. (10

It is probably also true that if Λ is a pseudo-lattice with $\Theta^{(2)}_\Lambda = E_4, E_6, E_{10}, E_{12}$, resp., then (103.5) still holds.

	E_4	E_6	E_{10}	E_{12}
"dimension" of pseudo lattice Λ_k whose theta series is E_k	8	12	20	24
dim $A_2 \otimes \Lambda$:	16	24	40	48
$2\left[\frac{\text{this}}{12}\right]+2$:	4	6	8	10
θ-series of extremal lattice	$1+O(q^4)$	$1+O(q^6)$	$1+O(q^8)$	$1+O(q^{10})$

Def of Λ_k

Now we ~~set~~ set $Z = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} z$ in the four basic Eisenstein series and expand in powers of z.

The Fourier series for the 4 Eisenstein series are given by Ozeki, extending earlier work of Maass, and Resnikoff & Saldaña : this is Ozeki–Washio (1982).

We write

$$\boxed{E_k(Z) = \sum_T a_k(T)\, e^{2\pi i \, tr(TZ)}} \qquad (104.1)$$

T runs thr' set Q_2 of symm mxs over $\frac{1}{2}\mathbb{Z}$ that are ≥ 0, and if T is of rank 2 then

$$a_k(T) = -\left(\frac{|2T|}{|d|}\right)^{k-\frac{3}{2}} \frac{4k}{|d|\, B_k\, B_{2k-2}} \prod_p c_p(k) \sum_{q=1}^{|d|-1} \left(\frac{d}{q}\right) (q+|d|B)^{k-1}$$

where B_k

d

$(q+($

c_p

p =

ε

~~S_p~~

S_p

but the

further

Only the

In (104.1)

if T =

trac

$\boxed{\text{Def of } \Lambda_k}$

are

…ashio (1982).

(104.1)

…ver $\frac{1}{2}\mathbb{Z}$

where B_k = Bernoulli #

d = discriminant of $\mathbb{Q}(\sqrt{-2|T|})$

$$(q+|d|B)^{k-1} = \sum_{n=0}^{k-1} \binom{k-1}{n} q^{k-1-n} |d|^n B_n$$

$$c_p(k) = S_p \frac{(1-\epsilon p^{1-k})}{(1-p^{-k})(1-p^{2-2k})}$$

p = any prime dividing $\frac{2|T|}{|d|}$

$\epsilon = \left(\frac{d}{p}\right)$

$$\cancel{S_p = \frac{(1-p^{-k})(1-p^{2-2k})}{(1-\epsilon p^{1-k})}}$$

S_p = p-adic density determined by T

but this is followed by pages of further formulae, giving $c_p(k)$

Only the genus of T matters.

In (104.1) we put $Z = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} z$, and then, if $T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$,

$$\operatorname{trace} TZ = \operatorname{tr}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} z = 2(a-b+c)z$$

and so, if we expand (101.4) in powers of $q = e^{2\pi i z}$, we get

(106.1) $$\Theta_{A_2\otimes\Lambda_k} = \sum_{m=0}^{\infty} C^{(k)}_m q^m$$ Λ_k = pseudo-lattice for E_k, $k=4,6,10,12$

then $$C^{(k)}_m = \sum_{T \text{ such that}} \alpha_k(T)$$

$$T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ with } (a-b+c) = \frac{m}{2}$$

and
$$\begin{cases} ac-b^2 \geq 0 & a,b,c\in\mathbb{Z} \\ a\equiv c\equiv 0\ (2) & \\ a,c\geq 0 & \\ a\leq c & |b|\leq a/2,\ c/2 \end{cases}$$

This gives the following T's for each m

$m=0$: $a-b+c=0$ $\therefore ac-(a+c)^2 \geq 0$

$\therefore a=b=c=0$ $\therefore T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$m=2$: $\left.\begin{array}{l} a-b+c=1 \\ ac-(a+c-1)^2\geq 0 \end{array}\right\}$ no solns

$m=4$: $a=0 \Rightarrow c=2 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$a=2 \Rightarrow c=0$ or $2 \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$a>2 \Rightarrow$ ✗. All 3 are equiv (as bin. q. f.'s) to $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ $\therefore 3\times\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$m=6$: (

$m=8$: 3

$m=10$: 3

$m=12$: 3x

The contri... follows: ... these sin... 1-dim E...

$1\times\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$3\times\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$3\times\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$

$3\times\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$

then the

s of $q = e^{2\pi i z}$,

Λ_k = pseudo-lattice for E_k

$= \frac{m}{2}$

≥ 0

ns

ex bin. q.f.'s)

$3 \times \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$\underline{m=6}$: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $(\Delta = 3)$

$m=8$: $3 \times \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$, $3 \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$(\Delta = 0)$ $(\Delta = 4)$

$m=10$: $3 \times \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $3 \times \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$(\Delta = 3)$ $(\Delta = 7)$

$m=12$: $3 \times \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$, $3 \times \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $6 \times \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $1 \times \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

The contributions to the four series (106.1) are as follows: the singular case first: these simply come from the expansions of ~~the~~ the 1-dim Eisenstein series

(A013974) (A3716 A298

		4	6	10	12
$1 \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	% 1	1	1	1	1
$3 \times \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	% q^4	240	−504	−264	$\frac{65520}{691}$
$3 \times \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$	% q^8	2160	−16632	−135432	$\frac{134250480}{691}$
$3 \times \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$	% q^{12}	6720	−122976	−5196576	$\frac{11606736960}{691}$

Then the rest we get from Ozeki-Washio :-

T_1	$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	13440	44352	$\frac{1136224320}{\cancel{8}219335}$	$\frac{244935250560}{590468483}$
T_2	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	30240	166320	$\frac{13130132400}{''}$	$\frac{5024786326560}{''}$
T_3	$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	138240	2128896	$\frac{1530879989760}{''}$	$\frac{1791551107906560}{''}$
T_4	$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$	181440	3792096	$\frac{4754093873760}{''}$	$\frac{7276749730810560}{''}$
$T_{\cancel{7}}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$	604800	24881472	$\frac{149800950573120}{''}$ (?)	$\frac{514419136607376000}{''}$

$\binom{42}{24}$ 604800 248881472 1498009505573120 " 5144191366073376000 "

So the q-expansions begin thus:

	$F_4 = A_2 \otimes \Lambda_4$	$F_6 = A_2 \otimes \Lambda_6$	$F_{10} = A_2 \otimes \Lambda_{10}$	$F_{12} = A_2 \otimes \Lambda_{12}$
q^0	1	1	1	1
q^2	0	0	0	0
q^4	720	-1512	-792	$\frac{196560}{691}$
q^6	13440	44352	~~[illegible]~~ ?	...

... or in Maple format :-

```
#eisenstein series 4,6,10,12 ;
array(1 .. 4,[(1)=series(1+720*q^4+13440*q^6+97200*q^8+455040*q^10+    F4
1714320*q^12+O(q^14),q,14),
(2)=series(1-1512*q^4+44352*q^6+449064*q^8+6519744*q^10+    F6
47263608*q^12+O(q^14),q,14),(3)=series(1-792*q^4+227244864/43867*q^6-    F10
9944907192/43867*q^8+919209728448/43867*q^10+34981193422296/43867*q^12+O(q^14)
,q,14),(4)=series(1+196560/691*q^4+22266840960/53678953*q^6+32657336384400/
53678953*q^8+488671648133760/53678953*q^10+53439465983183280/53678953*q^12+
O(q^14),q,14)])    — F12

eisenstein series 4,6,10,12 part 2;
x = phi_0(z), y = Delta_12(z);
series are evaluated (as in previous message) at Gram(A_2)*z.

[
x^8-48*x^2*y, ✓
x^12-72*x^6*y-1728*y^2, ✓
x^20-120*x^14*y+1728*x^8*y^2+1330587648/43867*x^2*y^3, ✓
x^24-144*x^18*y+3480192/691*x^12*y^2-2037901234176/53678953*x^6*y^3
   +21009383424000/53678953*y^4 ✓
]
```

(109.2)

— that fragment saved in fooRains circa line 4793 dated June 8 1998.

On the other hand we know ~~that~~ from "Mod 4" that the ring of theta series of even 3-modular lattices is

$$\mathbb{C}\left[\phi_0(z), \Delta_{12}(z)\right]$$

ϕ_0: dim 2 (A_2)
Δ_{12}: dim 12 (K_{12})

where $\Delta_{12}(z) = \left(\eta(z)\,\eta(3z)\right)^6$

Set (temporary notation)

$$x = \phi_0(z), \quad y = \Delta_{12}(z)$$

Then $F_4 = \phi_0^8 - 48\,\phi_0^2\,\Delta_{12}^1$, see p. 109, (109.2)

$$= x^8 - 48x^2 y$$

$\left.\begin{matrix} F_6 \\ F_{10} \\ F_{12} \end{matrix}\right\}$ see (109.2).

In other words,

$E_4^{(2)}$ @ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} z$ is $F_4 = \phi_0^8 - 48\phi_0^2\Delta_{12}$

$E_6^{(2)}$ @ " is $F_6 = \phi_0^{12} - 72\phi_0^6\Delta_{12} - 1728\,\Delta_{12}^2$

$E_8^{(2)}$ @ " is

$F_{10} =$

$F_{12} = \phi_0^2$

+

Mod[4]
3-modular
ϕ_0: dim 2 (A_2)
Δ_{12}: dim 12 (K_{12})

(8)

109, (109.2)

$8\,\Delta_{12}^2$

$E^{(2)}_{10}\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)_8$ is

$$F_{10} = \phi_0^{20} - 120\,\phi_0^{14}\Delta_{12} + 1728\,\phi_0^{8}\Delta_{12}^2 + \frac{1330587648}{43867}\,\phi_0^{2}\Delta_{12}^3$$

$E^{(2)}_{12}\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)_8$ is

$$F_{12} = \phi_0^{24} - 144\,\phi_0^{18}\Delta_{12} + \frac{3480192}{691}\,\phi_0^{12}\Delta_{12}^2 - \frac{2037901234176}{53678953}\,\phi_0^{6}\Delta_{12}^3 + \frac{21009383424000}{53678953}\,\Delta_{12}^4$$

We can clean up the last two,

since $$F_{10} - F_4 F_6 \propto x^2 y^3 =: \chi_{10} \quad \text{(say)}$$

$$F_{12} \bmod F_4^3 \,\&\, F_6^2 \Rightarrow y^3(x^6 + 12y) =: \chi_{12}$$

So now we have 4 polys in 2 variables:

rank 4 $\quad a = x^8 - 48x^2y = F_4$

6 $\quad b = x^{12} - 72x^6y - 1728y^2 = F_6$

10 $\quad c = x^2y^3 = \chi_{10}$

12 $\quad d = y^3(x^6 + 12y) = \chi_{12}$

Let us make the following substitutions:

$$12y \to Y \to z^6 \to z^3$$
$$x \to \sqrt{x}$$

and we get the equivalent set (but nicer)

rank 4: $a = x(x^3 - 4z^3)$

6: $b = x^6 - 6x^3z^3 - 12z^6$

10: $c = xz^9$

12 $d = z^9(x^3 + z^3)$

I take a, d as alg indep, and b, c as transients.

Using 'expres' in groupdir, I verified that the Molien series for this ring has the structure

$$\boxed{\frac{1, b, c, b^2, bc, c^2, b^3, b^2c, bc^2}{a, d}} \qquad (112.1)$$

meaning
is indep
there ar

C

Hence we
theorem 113
form of
theorem

then $\Theta^{(2}$

where

(say)

$=: \chi_{12}$

meaning that ~~the~~ each numerator term is indep of the preceding terms, and that there are syzygys for

$$c^3,\ b^4,\ b^3c,\ \text{and } b^2c^2.$$

Hence we have

<u>Theorem 113·1.</u> If $\Theta^{(2)}$ is a Siegel modular form of rank 2 — meaning that Igusa's theorem says it is an element of

$$\mathbb{C}\left[E_4^{(2)}, E_6^{(2)}, E_{10}^{(2)}, E_{12}^{(2)}\right] —$$

then $\Theta^{(2)}\left(\begin{pmatrix}2&1\\1&2\end{pmatrix}z\right)$ is in the ring (112·1),

where
$$a = F_4 = E_4^{(2)} @ \begin{pmatrix}2&1\\1&2\end{pmatrix}z = \phi_0^8 - 48\phi_0^2\Delta_{12}$$
$$b = F_6 = E_6^{(2)} @ \text{ " } = \phi_0^{12} - 72\phi_0^6\Delta_{12} - 1728\Delta_{12}^2$$
$$c = \chi_{10} = x^2y^3 = \phi_0(z)^2\Delta_{12}(z)^3$$
$$d = \chi_{12} = y^3(x^6 + 12y) = \cancel{\phi_0 \ldots}$$
$$= \Delta_{12}(z)^3\left(\phi_0(z)^6 + 12\Delta_{12}(z)\right).$$

ed
has the

$\boxed{bc^2}$

Cor (114.1) If Λ is an even unimodular lattice then $A_2 \otimes \Lambda$ has theta series which is an element of the ring

$$(114.2) \qquad \frac{1,\ b^2,\ bc,\ c^2}{a,\ b}.$$

Proof We have to restrict (112.1) to ranks (or dts!!) that are multiples of 4 (ie so dim is a multiple of (6))

So we have to replace $\{b, c\}$ by $\{b^2, bc, c^2\}$.

One now verifies using 'expres' that (114.2) is correct [In partic, that the required syz. for b^4, b^2c^2 and c^4 exist] ✓. □

Example Theta series of $A_2 \otimes E_8$ is

$$a = F_4 = \phi_0^8 - 48\,\phi_0^2\,\Delta_{12}$$

$$(A033700) \qquad = 1 + 720q^4 + 13440q^6 + \cdots$$

6/13/98

Next What can be said about theta series of $A_2 \otimes \Lambda$ if Λ is an odd unimod. lattice?

Q: (114.3) Is there any analog of Igusa's result?

Without that it will be difficult. On the other hand the evidence suggests that following is true:

Conjecture
lattice A
Then (6)

with the sy

The evidence
series for
Examples :
o A
o A
o A

o theta series of A

o " " "

lattice
hich is

(or ...s!)

multiple

$\{b^2, bc, c^2\}$.
(114.2)

✓. □

$40q^6 + \cdots$

6/13/98

od. lattice?
result?

hand the

<u>Conjecture 115.1</u> Let Θ be theta series of any lattice $A_2 \otimes \Lambda$, Λ = unimodular lattice. Then Θ belongs to the ring

$$\frac{1,\ \phi_0^2\Delta_{12}}{\phi_0,\ \Delta_{12}^2} \qquad (115.2)$$

with the syzygy $(\phi_0^2\Delta_{12})^2 = \phi_0^4 \cdot \Delta_{12}^2$.

<u>The evidence</u> is that this explains all the theta series for $\dim \Lambda \le 15$.

Examples:

- $A_2 \otimes \mathbb{Z} \Rightarrow \phi_0$
- $A_2 \otimes$ even unimod $\Rightarrow \phi_0^2\Delta_{12}$ & Δ_{12}^2, by page 110
- $A_2 \otimes D_{12}^+$ has θ series
  $$\phi_0^{12} - 72\phi_0^6\Delta_{12} + 576\Delta_{12}^2$$
- theta series of $A_2 \otimes E_7^{2+}$ is
  $$\phi_0^{14} - 84\phi_0^8\Delta_{12} + 1008\phi_0^2\Delta_{12}^2$$
- " " " $A_2 \otimes A_{15}^+$ is
  $$\phi_0^{15} - 90\phi_0^9\Delta_{12} + 1260\phi_0^3\Delta_{12}^2.$$