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Let $g_k(s_k)$ denote the largest (smallest) prime divisor of the natural number $k \geq 2$ and let $g_1 = s_1 = 1$.

Recently J. van de Lune raised the problem of determining the asymptotic behaviour of the sums

$$\sum_{k\leq x} g_k \quad \text{and} \quad \sum_{k\leq x} s_k .$$

Van de Lune and E. Wattel showed that the upper and lower limits of

$$\frac{\log x}{x^2} \cdot \sum_{k\leq x} g_k$$

lie between ½ and 1. In this report the following theorem will be proved.

THEOREM. (i) $$\sum_{k\leq x} \ell_k = \frac{\pi^2}{12}\frac{x^2}{\log x} + O(x^2 \log^{-3/2} x \text{ loglog } x),$$

(ii) $$\sum_{k\leq x} s_k = \frac{1}{2}\frac{x^2}{\log x} + O(x^2 \log^{-2} x).$$

PROOF.

(i) First $\sum_{k\leq x} \ell_k \leq \sum_{p\leq x} [\frac{x}{p}]p$, because the last sum contains for each k all prime factors of k instead of only the largest one. Also $\sum_{k\leq x} \ell_k \geq \sum_{\sqrt{x}<p\leq x} [\frac{x}{p}]\, p$ since $p>\sqrt{x}$, $k\leq x$, $p|k$ imply $p = \ell_k$. Clearly $\sum_{p\leq\sqrt{x}} [\frac{x}{p}]\cdot p \leq x\sqrt{x}$, and therefore

(1) $$\sum_{k\leq x} \ell_k = \sum_{p\leq x} [\frac{x}{p}]\, p + O(x^{3/2}).$$

Let f be an increasing function such that:

(a) $\lim_{x\to\infty} f(x) = \infty$

(b) $\exists_{\varepsilon>0}: f(x) = O(x^{1-\varepsilon})$.

Then

$$\sum_{p\leq x/f(x)} [\frac{x}{p}]\, p \leq x\, \pi(\frac{x}{f(x)}) ,$$

and since $\pi(x) = O(\frac{x}{\log x})$ it follows that

$$\sum_{p\le x}\left[\frac{x}{p}\right]p = \sum_{x/f(x)\le p\le x}\left[\frac{x}{p}\right]p + O\left(\frac{x^2}{\log x\cdot f(x)}\right) =$$

see (ii)

$$= \sum_{n=1}^{f(x)}\sum_{p\le\frac{x}{n}} p + O\left(\frac{x^2}{\log x\cdot f(x)}\right) =$$

$$= \sum_{n=1}^{f(x)}\left(\frac{(\frac{x}{n})^2}{2\log\frac{x}{n}} + O\left(\frac{x^2}{\log^2 x}\right)\right) + O\left(\frac{x^2}{\log x\cdot f(x)}\right) =$$

$$= \frac{x^2}{2\log x}\sum_{n=1}^{f(x)}\frac{1}{n^2}\left(1-\frac{\log n}{\log x}\right)^{-1} + O\left(\frac{x^2 f(x)}{\log^2 x}\right) + O\left(\frac{x^2}{\log x\cdot f(x)}\right) =$$

$$= \frac{x^2}{2\log x}\left(\frac{\pi^2}{6} + O\left(\frac{1}{f(x)}\right)\right)\left(1 + O\left(\frac{f(x)\log f(x)}{\log x}\right)\right) +$$

$$+ O\left(\frac{x^2 f(x)}{\log^2 x}\right) + O\left(\frac{x^2}{\log x\cdot f(x)}\right) =$$

$$= \frac{\pi^2}{6}\cdot\frac{x^2}{2\log x} + O\left(\frac{x^2 f(x)\log f(x)}{\log^2 x}\right) + O\left(\frac{x^2}{\log x\cdot f(x)}\right).$$

Now take $f(x) = \log^{\frac{1}{2}} x$. Then

$$\frac{x^2 f(x)\log f(x)}{\log^2 x} = \frac{x^2\log\log x}{2\log^{3/2} x} \quad\text{and}\quad \frac{x^2}{\log x\cdot f(x)} = \frac{x^2}{\log^{3/2} x}$$

which proves (i).

(ii) First $\sum_{k\le x} s_k \ge \sum_{p\le x} p$; also

$$\sum_{k\le x} s_k \le \sum_{p\le x} p + \sum_{k\le x}\sqrt{k} \le \sum_{p\le x} p + x^{3/2}.$$

Thus

$$(2)\qquad \sum_{k\le x} s_k = \sum_{p\le x} p + O(x^{3/2}).$$

Now

$$\sum_{p\le x} p = \int_{2^-}^{x} x \, d\pi(x) = \int_{2^-}^{x} \left(\frac{x}{\log x} + 0(\frac{x}{\log^2 x})\right) dx =$$

$$= \frac{x^2}{2\log x} + 0\left(\frac{x^2}{\log^2 x}\right)$$

which proves (ii).