On the partial sums of the Zeta function $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for $0 < s < 1$

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Let $0 < s < 1$ and $x \geq 1$ be two real numbers, and let $n \geq 1$ be an integer. Define $S_s(n) = \sum_{k=1}^{n} \frac{1}{k^s}$ and denote by $n_s(x)$ the smallest integer such that $S_s(n_s(x)) \geq x$. Set $x_s = ((1-s)(x-\gamma_s)+1)^{\frac{1}{1-s}}$, where γ_s is the analogue of the Euler-Mascheroni constant, i.e.,

$$\gamma_s = \lim_{m\to\infty}\left(-\int_1^m \frac{1}{t^s}dt + \sum_{k=1}^{m}\frac{1}{k^s}\right).$$

We claim that

$$n_s(x) \in \{\lfloor x_s \rfloor - 1, \lfloor x_s \rfloor, \lfloor x_s \rfloor + 1\}^1,$$

Indeed[2], setting $n = n_s(x)$, by the Euler-MacLaurin formula, together with the second mean-value theorem (e.g., [3, 23.1.30] and [4]), we have

$$\int_1^n \frac{1}{t^s}dt + \gamma_s + \frac{1}{2n^s} \leq S_s(n) \leq \int_1^n \frac{1}{t^s}dt + \gamma_s + \frac{1}{2n^s} + \frac{s}{8n^{1+s}}.$$

It follows that

$$\begin{aligned}\frac{(n-1)^{1-s}}{1-s} + \frac{1}{2(n-1)^s} &\leq S_s(n-1) < x - \gamma_s + \frac{1}{1-s} \\ &\leq S_s(n) \leq \frac{n^{1-s}}{1-s} + \frac{1}{2n^s} - \frac{7s}{n^{s+1}}.\end{aligned}$$

Thus,

$$n - 1 < ((1-s)(x-\gamma_s)+1)^{\frac{1}{1-s}} \leq n\left(1 + \frac{1-s}{2n}\right)^{\frac{1}{1-s}}.$$

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[1]Notice that Boas [1] proves a stronger result in a more general setting. However, his results hold for "sufficiently large" x and he also uses $o(1)$ notation.

[2]Our proof is almost identical to the one provided by Comtet [2].

By a variant of Bernoulli's inequality (e.g., [5, (r_5')]), we have

$$n\left(1+\frac{1-s}{2n}\right)^{\frac{1}{1-s}} \le n+\frac{1}{2}+\frac{1}{n},$$

leading to

$$((1-s)(x-\gamma_s)+1)^{\frac{1}{1-s}}-\frac{1}{2}-\frac{1}{n} \le n < ((1-s)(x-\gamma_s)+1)^{\frac{1}{1-s}}+1,$$

from which the assertion immediately follows.

Remark 1. Related sequences: A054040 and A231405.

References

[1] Boas R. Growth of partial sums of divergent series. Mathematics of Computation. 1977;31(137):257-64.

[2] Comtet L. 5346. The American Mathematical Monthly. 1967;74(2):209-9.

[3] Abramowitz M, Stegun IA. Handbook of mathematical functions with formulas, graphs, and mathematical tables. vol. 55. US Government printing office; 1948.

[4] Hobson EW. On the second mean-value theorem of the integral calculus. Proceedings of the London Mathematical Society. 1909;2(1):14-23.

[5] Li YC, Yeh CC. Some equivalent forms of Bernoulli's inequality: A survey. Applied Mathematics. 2013;4(07):1070.