Number of binary matrices with fixed number of unit columns up to row and column permutations

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Define a unit column of a binary matrix to be a column with only one 1.

Let $b_{m,n,k}$ be the number of m x n binary matrices with $k=0, 1, ..., n$ unit columns, up to row and column permutations. Denote by $Z(E_2^{S_m}; x_1, x_2, ...; t_1, t_2, ...)$ modified cycle index of power group $E_2^{S_m}$ and let $f_m(x,t)$ be obtained by replacing x_i with $\dfrac{1}{1-x^i}$ and t_i with $\dfrac{1}{1-x^i t^i}$. If

$$F_{m,k}(x,t) = \frac{1}{k!}\frac{\partial^k}{\partial t^k} f_m(x,t) \text{ and } g_{m,k}(x) = F_{m,k}(x,0) \text{ then } b_{m,n,k} \text{ is the coefficient of } x^n \text{ in } g_{m,k}(x).$$

Cycle index $Z(E_2^{S_m}; x_1, x_2, ...; t_1, t_2, ...)$ can be calculated in the folowing way:

$$Z(E_2^{S_m}; x_1, x_2, ...; t_1, t_2, ...) = \frac{1}{m!}\sum_{\pi(m)}\frac{m!}{k_1!1^{k_1}k_2!2^{k_2}...k_m!m^{k_m}} \cdot \prod_{i|k} x_i^{l_i-k_i} t_i^{k_i}, \text{ where } \pi(m) \text{ runs through all}$$

partitions of m (i.e. nonnegative solutions of $k_1 + 2k_2 + ... + mk_m = m$);

$k = lcm\{i \mid k_i \neq 0\}$;

$$l_i = l_i(\pi) = \frac{1}{i}\sum_{d|i}\mu(\frac{i}{d}) \cdot 2^{\sum_{j=1}^{m}(j,d)k_j}, \text{ where } \mu \text{ is Mobius function and } (j,d)=gcd\{j,d\}.$$

Modified cycle indices $Z(E_2^{S_m}; x_1, x_2,\dots; t_1, t_2,\dots)$ of power group $E_2^{S_m}$,
m=1,2,…,6

Explanation:

$(1^\wedge k_1 * 2^\wedge k_2 * \dots + 1^\wedge l_1 * 2^\wedge l_2 * \dots)$ $means$ $x_1^{k_1} x_2^{k_2} \dots t_1^{l_1} t_2^{l_2} \dots$.

Exempla:

$(1\wedge3*2\wedge1+1\wedge1*2\wedge1)= x_1^3 x_2 t_1 t_2$,

$1/6*(1*(1\wedge5+1\wedge3)+3*(1\wedge3*2\wedge1+1\wedge1*2\wedge1)+2*(1\wedge2*3\wedge1+3\wedge1))= \dfrac{1}{6}(x_1^5 t_1^3 + 3 x_1^3 x_2 t_1 t_2 + 2 x_1^2 x_3 t_3)$.

$1/1*(1*(1\wedge1+1\wedge1))$

$1/2*(1*(1\wedge2+1\wedge2)+1*(1\wedge2+2\wedge1))$

$1/6*(1*(1\wedge5+1\wedge3)+3*(1\wedge3*2\wedge1+1\wedge1*2\wedge1)+2*(1\wedge2*3\wedge1+3\wedge1))$

$1/24*(1*(1\wedge12+1\wedge4)+6*(1\wedge6*2\wedge3+1\wedge2*2\wedge1)+3*(1\wedge4*2\wedge4+2\wedge2)+8*(1\wedge3*3\wedge3+1\wedge1*3\wedge1)+6*(1\wedge2*2\wedge1*4\wedge2+4\wedge1))$

$1/120*(1*(1\wedge27+1\wedge5)+10*(1\wedge13*2\wedge7+1\wedge3*2\wedge1)+15*(1\wedge7*2\wedge10+1\wedge1*2\wedge2)+20*(1\wedge6*3\wedge7+1\wedge2*3\wedge1)+20*(1\wedge4*2\wedge1*3\wedge3*6\wedge2+2\wedge1*3\wedge1)+30*(1\wedge3*2\wedge2*4\wedge5+1\wedge1*4\wedge1)+24*(1\wedge2*5\wedge5+5\wedge1))$

$1/720*(1*(1\wedge58+1\wedge6)+15*(1\wedge28*2\wedge15+1\wedge4*2\wedge1)+45*(1\wedge14*2\wedge22+1\wedge2*2\wedge2)+15*(1\wedge8*2\wedge25+2\wedge3)+40*(1\wedge13*3\wedge15+1\wedge3*3\wedge1)+120*(1\wedge7*2\wedge3*3\wedge7*6\wedge4+1\wedge1*2\wedge1*3\wedge1)+40*(1\wedge4*3\wedge18+3\wedge2)+90*(1\wedge6*2\wedge4*4\wedge11+1\wedge2*4\wedge1)+90*(1\wedge4*2\wedge5*4\wedge11+2\wedge1*4\wedge1)+144*(1\wedge3*5\wedge11+1\wedge1*5\wedge1)+120*(1\wedge2*2\wedge1*3\wedge2*6\wedge8+6\wedge1))$

Generating functions $f_m(x,t)$, m=1,2,…,6

1/1*(1/(1-x^1)^1/(1-x^1*t^1)^1)

1/2*(1/(1-x^1)^2/(1-x^1*t^1)^2+1/(1-x^1)^2/(1-x^2*t^2)^1)

1/6*(1/(1-x^1)^5/(1-x^1*t^1)^3+3/(1-x^1)^3/(1-x^2)^1/(1-x^1*t^1)^1/(1-x^2*t^2)^1+2/(1-x^1)^2/(1-x^3)^1/(1-x^3*t^3)^1)

1/24*(1/(1-x^1)^12/(1-x^1*t^1)^4+6/(1-x^1)^6/(1-x^2)^3/(1-x^1*t^1)^2/(1-x^2*t^2)^1+3/(1-x^1)^4/(1-x^2)^4/(1-x^2*t^2)^2+8/(1-x^1)^3/(1-x^3)^3/(1-x^1*t^1)^1/(1-x^3*t^3)^1+6/(1-x^1)^2/(1-x^2)^1/(1-x^4)^2/(1-x^4*t^4)^1)

1/120*(1/(1-x^1)^27/(1-x^1*t^1)^5+10/(1-x^1)^13/(1-x^2)^7/(1-x^1*t^1)^3/(1-x^2*t^2)^1+15/(1-x^1)^7/(1-x^2)^10/(1-x^1*t^1)^1/(1-x^2*t^2)^2+20/(1-x^1)^6/(1-x^3)^7/(1-x^1*t^1)^2/(1-x^3*t^3)^1+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2/(1-x^2*t^2)^1/(1-x^3*t^3)^1+30/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5/(1-x^1*t^1)^1/(1-x^4*t^4)^1+24/(1-x^1)^2/(1-x^5)^5/(1-x^5*t^5)^1)

1/720*(1/(1-x^1)^58/(1-x^1*t^1)^6+15/(1-x^1)^28/(1-x^2)^15/(1-x^1*t^1)^4/(1-x^2*t^2)^1+45/(1-x^1)^14/(1-x^2)^22/(1-x^1*t^1)^2/(1-x^2*t^2)^2+15/(1-x^1)^8/(1-x^2)^25/(1-x^2*t^2)^3+40/(1-x^1)^13/(1-x^3)^15/(1-x^1*t^1)^3/(1-x^3*t^3)^1+120/(1-x^1)^7/(1-x^2)^3/(1-x^3)^7/(1-x^6)^4/(1-x^1*t^1)^1/(1-x^2*t^2)^1/(1-x^3*t^3)^1+40/(1-x^1)^4/(1-x^3)^18/(1-x^3*t^3)^2+90/(1-x^1)^6/(1-x^2)^4/(1-x^4)^11/(1-x^1*t^1)^2/(1-x^4*t^4)^1+90/(1-x^1)^4/(1-x^2)^5/(1-x^4)^11/(1-x^2*t^2)^1/(1-x^4*t^4)^1+144/(1-x^1)^3/(1-x^5)^11/(1-x^1*t^1)^1/(1-x^5*t^5)^1+120/(1-x^1)^2/(1-x^2)^1/(1-x^3)^2/(1-x^6)^8/(1-x^6*t^6)^1)

Generating functions $g_{3,k}(x)$, k=0,1,...,10

1/6*(1/(1-x^1)^5+3/(1-x^1)^3/(1-x^2)^1+2/(1-x^1)^2/(1-x^3)^1)

x/6*(3/(1-x^1)^5+3/(1-x^1)^3/(1-x^2)^1)

x^2/6*(6/(1-x^1)^5+6/(1-x^1)^3/(1-x^2)^1)

x^3/6*(10/(1-x^1)^5+6/(1-x^1)^3/(1-x^2)^1+2/(1-x^1)^2/(1-x^3)^1)

x^4/6*(15/(1-x^1)^5+9/(1-x^1)^3/(1-x^2)^1)

x^5/6*(21/(1-x^1)^5+9/(1-x^1)^3/(1-x^2)^1)

x^6/6*(28/(1-x^1)^5+12/(1-x^1)^3/(1-x^2)^1+2/(1-x^1)^2/(1-x^3)^1)

x^7/6*(36/(1-x^1)^5+12/(1-x^1)^3/(1-x^2)^1)

x^8/6*(45/(1-x^1)^5+15/(1-x^1)^3/(1-x^2)^1)

x^9/6*(55/(1-x^1)^5+15/(1-x^1)^3/(1-x^2)^1+2/(1-x^1)^2/(1-x^3)^1)

x^10/6*(66/(1-x^1)^5+18/(1-x^1)^3/(1-x^2)^1)

Generating functions $g_{4,k}(x)$, k=0,1,...,10

1/24*(1/(1-x^1)^12+6/(1-x^1)^6/(1-x^2)^3+3/(1-x^1)^4/(1-x^2)^4+8/(1-x^1)^3/(1-x^3)^3+6/(1-x^1)^2/(1-x^2)^1/(1-x^4)^2)

x/24*(4/(1-x^1)^12+12/(1-x^1)^6/(1-x^2)^3+8/(1-x^1)^3/(1-x^3)^3)

x^2/24*(10/(1-x^1)^12+24/(1-x^1)^6/(1-x^2)^3+6/(1-x^1)^4/(1-x^2)^4+8/(1-x^1)^3/(1-x^3)^3)

x^3/24*(20/(1-x^1)^12+36/(1-x^1)^6/(1-x^2)^3+16/(1-x^1)^3/(1-x^3)^3)

x^4/24*(35/(1-x^1)^12+54/(1-x^1)^6/(1-x^2)^3+9/(1-x^1)^4/(1-x^2)^4+16/(1-x^1)^3/(1-x^3)^3+6/(1-x^1)^2/(1-x^2)^1/(1-x^4)^2)

x^5/24*(56/(1-x^1)^12+72/(1-x^1)^6/(1-x^2)^3+16/(1-x^1)^3/(1-x^3)^3)

x^6/24*(84/(1-x^1)^12+96/(1-x^1)^6/(1-x^2)^3+12/(1-x^1)^4/(1-x^2)^4+24/(1-x^1)^3/(1-x^3)^3)

x^7/24*(120/(1-x^1)^12+120/(1-x^1)^6/(1-x^2)^3+24/(1-x^1)^3/(1-x^3)^3)

x^8/24*(165/(1-x^1)^12+150/(1-x^1)^6/(1-x^2)^3+15/(1-x^1)^4/(1-x^2)^4+24/(1-x^1)^3/(1-x^3)^3+6/(1-x^1)^2/(1-x^2)^1/(1-x^4)^2)

x^9/24*(220/(1-x^1)^12+180/(1-x^1)^6/(1-x^2)^3+32/(1-x^1)^3/(1-x^3)^3)

x^10/24*(286/(1-x^1)^12+216/(1-x^1)^6/(1-x^2)^3+18/(1-x^1)^4/(1-x^2)^4+32/(1-x^1)^3/(1-x^3)^3)

Generating functions $g_{5,k}(x)$, k=0,1,...,10

1/120*(1/(1-x^1)^27+10/(1-x^1)^13/(1-x^2)^7+15/(1-x^1)^7/(1-x^2)^10+20/(1-x^1)^6/(1-x^3)^7+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+30/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5+24/(1-x^1)^2/(1-x^5)^5)

x/120*(5/(1-x^1)^27+30/(1-x^1)^13/(1-x^2)^7+15/(1-x^1)^7/(1-x^2)^10+40/(1-x^1)^6/(1-x^3)^7+30/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^2/120*(15/(1-x^1)^27+70/(1-x^1)^13/(1-x^2)^7+45/(1-x^1)^7/(1-x^2)^10+60/(1-x^1)^6/(1-x^3)^7+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+30/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^3/120*(35/(1-x^1)^27+130/(1-x^1)^13/(1-x^2)^7+45/(1-x^1)^7/(1-x^2)^10+100/(1-x^1)^6/(1-x^3)^7+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+30/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^4/120*(70/(1-x^1)^27+220/(1-x^1)^13/(1-x^2)^7+90/(1-x^1)^7/(1-x^2)^10+140/(1-x^1)^6/(1-x^3)^7+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+60/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^5/120*(126/(1-x^1)^27+340/(1-x^1)^13/(1-x^2)^7+90/(1-x^1)^7/(1-x^2)^10+180/(1-x^1)^6/(1-x^3)^7+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+60/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5+24/(1-x^1)^2/(1-x^5)^5)

x^6/120*(210/(1-x^1)^27+500/(1-x^1)^13/(1-x^2)^7+150/(1-x^1)^7/(1-x^2)^10+240/(1-x^1)^6/(1-x^3)^7+40/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+60/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^7/120*(330/(1-x^1)^27+700/(1-x^1)^13/(1-x^2)^7+150/(1-x^1)^7/(1-x^2)^10+300/(1-x^1)^6/(1-x^3)^7+20/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+60/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^8/120*(495/(1-x^1)^27+950/(1-x^1)^13/(1-x^2)^7+225/(1-x^1)^7/(1-x^2)^10+360/(1-x^1)^6/(1-x^3)^7+40/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+90/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^9/120*(715/(1-x^1)^27+1250/(1-x^1)^13/(1-x^2)^7+225/(1-x^1)^7/(1-x^2)^10+440/(1-x^1)^6/(1-x^3)^7+40/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+90/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5)

x^10/120*(1001/(1-x^1)^27+1610/(1-x^1)^13/(1-x^2)^7+315/(1-x^1)^7/(1-x^2)^10+520/(1-x^1)^6/(1-x^3)^7+40/(1-x^1)^4/(1-x^2)^1/(1-x^3)^3/(1-x^6)^2+90/(1-x^1)^3/(1-x^2)^2/(1-x^4)^5+24/(1-x^1)^2/(1-x^5)^5)