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Music and ternary continued fractions.

By

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In «The American Mathematical Monthly» (T. 55, 545, 1948) J. M. BARBOUR has written a very interesting article on music and ternary continued fractions. He had to find integers x, y and z such that the double approximative equation

$$\frac{\log \frac{5}{4}}{x} \approx \frac{\log \frac{3}{2}}{y} \approx \frac{\log 2}{z}$$

is satisfied in the best manner. At first he used the algorithm of Jacobi. But as there are two serious faults with the results obtained in this way he modifies the method, introducing «reversed ternary continued fractions» and at last introduces what he calls «mixed ternary continued fractions».

It may be of interest to remark that the generalisation of the continued fractions which I have given in 1919 [1] directly gives the same numbers (and some more) as those found by Barbour.

My algorithm consists of the following rule:

I write the three given numbers a, b and c and the three triplets (1, 0, 0), (0, 1, 0) and (0, 0, 1) in the following manner:

$$\begin{matrix} a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{matrix}$$

Then I replace the greatest of the three numbers a, b, c by the difference between the greatest and the midmost of them. Likewise I replace the triplet belonging to the midmost of the three numbers a, b, c by a triplet made by addition of the numbers in the two triplets belonging to the greatest and the midmost of a, b, c. The other numbers are only to be reproduced. If $a > b > c$, we get the new scheme

$a-b$	1	0	0
b	1	1	0
c	0	0	1

The process is to be continued in the same way. In the case treated by Barbour the first steps of calculation by my algorithm will be:

0,30103	1	0	0
0,17609	0	1	0
0,09691	0	0	1
0,12494	1	0	0
0,17609	1	1	0
0,09691	0	0	1
0,12494	2	1	0
0,05115	1	1	0
0,09691	0	0	1
0,02803	2	1	0
0,05115	1	1	0
0,09691	2	1	1

The triplets (z, y, x) then will be

2	1	1
3	2	1
5	3	2
7	4	2
12	7	4
19	11	6
31	18	10
34	20	11
53	31	17
87	51	28
118	69	38
205	*120*	*66*
323	*189*	*104*
441	*258*	*142*
259	327	180
612	358	197
730	427	235

It is remarkable that these numbers are the same as those found by Barbour, by his much more complicated method. Only the triplets (205, 120, 66) and (323, 189, 104) and (441, 258, 142) are not to be found in the series of Barbour.

It is also interesting that, as Barbour remarks, the method of Jacobi here fails to give valid results. Stieltjes and Hermite also observed that the algorithm of Jacobi is defective. I quote here some lines of their correspondence [2]:

«L'algorithme de Jacobi qui a appelé votre attention s'offre immédiatement à l'ésprit, c'est l'extension toute naturelle, à trois nombres, du procédé élémentaire par la recherche du plus grand commun diviseur entre deux nombres. Mais je crois, dans cette circonstance, l'analogie trompeuse». (Hermite, 1889).

«Vous m'avez convaincu complètement qu'il faut abandonner l'analogie qui conduit à l'algorithme de Jacobi». En étudiant, au point de vue arithmétique, l'algorithme de Jacobi, j'avais vu aussi se présenter bien des difficultés; ainsi par exemple, dans le cas des fractions continues, la série des quotients incomplets (qui n'est assujettie à aucune condition) détermine toujours un rapport déterminé, mais il n'en est pas ainsi dans l'algorithme de Jacobi. C'est là une circonstance bien embarrasante et qui avait déjà beaucoup ébranlé ma confiance». (Stieltjes, 1890).

«La recherche des fractions $\frac{P'}{P}, \frac{P''}{P}$ qui approchent le plus de deux nombres donnés n'a cessé, depuis plus de 50 ans, de me préoccuper et aussi de me désespérer». (Hermite, 1894).

In my opinion, all generalisations of continued fractions which make use of *division* are foredoomed to failure. It is interesting to observe that the original form of the algorithm of Euclid [3] also uses *subtraction* and not division. For generalisations this distinction is a matter of great importance.

[1] Viggo Brun: En generalisation av kjedebrøken (avec un résumé en français) Videnskapsselskapets skrifter, Oslo 1919 and 1920. Compare also Nils Pipping: Arithm. Kriterien für reelle alg. Zahlen, Acta Acad. Aboensis 1933.

[2] Correspondance d'Hermite et de Stieltjes, Tome II (p. 11, 23).

[3] Euclid: Elements, Book VII, 2 and Book X, 3.

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