Simple proofs of some facts related to the Bell sequence and triangles A007318 (Pascal) and A071919 (enlarged Pascal)

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Abstract

It is known from *Bernstein* and *Sloane* 1995 [1], that the BINOMIAL (also known as (*aka*) *Pascal*) transform of the *Bell* sequence A000110 $\{B(n)\}_0^\infty$ is the shifted sequence $\{B(n+1)\}_0^\infty$. Here another proof by elementary means is given. This fact implies that the *Bell* sequence is an eigensequence to the enlarged *Pascal* matrix A071918 with eigenvalue 1; *i.e.*, a fixed point under iteration.

It is also proved that the nth power of the lower triangular matrix A071918 in the limit $n \to \infty$ has in its first column the *Bell* sequence and zeros otherwise. This shows that the *Bell* sequence is the asymptotic vector for the initial vector $(1,0,0,0,...)^\top$ under iteration. This fact is tied to the property α of the *Bell* sequence stated also in *Bernstein* and *Sloane* [1].

1 Introduction

Motivated by e-mail correspondence with Gary W. Adamson the author was led to provide proofs for several of his conjectures concerning the *Bell* sequence A000110, the *Pascal* matrix A007318 and the corresponding matrix A071919 which has the main diagonal $[1,0,0,0,...]$ on top of *Pascal*'s triangle when written as lower triangular array. Neither originality nor priority is claimed. In fact, proposition 1 appears as special instance in *Bernstein* and *Sloane* [1]. All proofs involve elementary operations on lower triangular number matrices, based on the knowledge of the ordinary generating function ($o.g.f.$) of the sequence of the mth column. All considered triangles $T(n,m)$ have offset $[0,0]$, i.e., $n \geq 0$ and $m \geq 0$, and the triangularity condition is $T(n,m) = 0$ if $n < m$. All manipulations are formal. No convergence issues are considered. This means especially that infinite sums can be interchanged.

2 Bell sequence as eigensequence of A071919

Proposition 1. *[1, Bernstein and Sloane 1995]* $B(n+1) = \sum_{m=0}^{n} P(n,m)\,B(m)$, *with the lower triangular (infinite dimensional) Pascal matrix* $P(n,m) =$ *A007318*$(n,m) := \binom{n}{m}$ *and the Bell sequence A000110.*

Definition 2. The BINOMIAL transform of a sequence C with members $C(n)$ is the sequence D with members $D(n) = \sum_{m=0}^{n} P(n,m)\,C(m)$, $n \geq 0$. In matrix notation $D = \mathbf{P}\,C$. Because of the *Pascal* matrix $\mathbf{P}$ appearance, D could also be called *Pascal* or $\mathbf{P}$ transform of C.

Therefore, proposition 1 states the result for the BINOMIAL transform of the *Bell* sequence, known from *Bernstein* and *Sloane* [1]. In the notation of this reference it would be written as $\mathbf{P} \circ B = L \circ B$ with the shift operation $L \circ [B_0, B_1, B_2, ...] = [B_1, B_2, B_3, ...]$.

Definition 3. The sequence D with members $D(n) = \sum_{m=0}^{\infty} T(n,m)\,C(m)$, $m \geq 0$, is called the $\mathbf{T}-$transform of the sequence C with members $C(m)$, $m \geq 0$. Here $\mathbf{T}$ is any lower triangular matrix.

Corollary 4. $\displaystyle\sum_{m=0}^{n} T(n,m)\,B(m) = B(n)$ *with the lower triangular (infinite dimensional) enlarged Pascal matrix* $T(n,m) =$ *A071919. Therefore, the Bell sequence is an eigensequence to the matrix A071919 with eigenvalue* 1. *In matrix notation* $B = \mathbf{T}\,B$, *and* B *is a fixed point, in an infinite dimensional* $\mathbb{R}$ *vector space, under the* $\mathbf{T}$ *transformation.*

Proof. Four lemmata and a definition of the *Bell* numbers are first given.

Lemma 5. *The Pascal lower triangular matrix is the Riordan array (or triangle)* $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$, *which means that the o.g.f. of the sequence in column nr.* m *is*

$$P_m(x) = \frac{1}{1-x}\left(\frac{x}{1-x}\right)^m, \; m \geq 0. \tag{1}$$

The proof of this lemma is obvious from the (ordinary) convolution property of *Pascal*'s triangle. For the notion of *Riordan* matrices see the paper by *Shapiro et al.* [5].

Lemma 6. *The Stirling triangle of the second kind,* $\mathbf{S} =$ *A048993 (with first column* $(1,0,0,...)^\top$*), has as o.g.f. of its mth column sequence*

$$S_m(x) = \frac{x^m}{\prod_{j=1}^{m}(1 - j\,x)}, \; m \geq 0. \tag{2}$$

This lemma appears as theorem C on p. 207 of *Comtet*'s book [2]. Its proof is also given there and it is based on the partial fraction decomposition of the *o.g.f.* and an explicit formula for $S(n,m)$, the *Stirling* numbers of the second kind. This formula is due to their subset number property. See theorem A on p. 204 of *Comtet*'s book.

Definition 7. The *Bell* numbers $B(n)$, $n \geq 0$ are the row sums of the *Stirling* triangle of the second kind:

$$B(n) \;:=\; \sum_{m=0}^{n} S(n,m)\ . \tag{3}$$

Lemma 8. *The o.g.f. of the Bell numbers is*

$$\mathcal{B}(x) \;:=\; \sum_{n=0}^{\infty} B(n)\, x^n \;=\; \sum_{m=0}^{\infty} \frac{x^m}{\prod_{j=1}^{m} (1 - j\, x)}\,. \tag{4}$$

For $m = 0$ the product has to be replaced by 1.

Proof. This lemma has been included as a comment by R. Stephan under A000110 without a proof. The proof uses definition 7 and lemma 6.

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} S(n,m)\, x^n \text{ , interchanging the summations yields:}$$

$$= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} S(n,m)\, x^n \;=\; \sum_{m=0}^{\infty} S_m(x) \;=\; \sum_{m=0}^{\infty} \frac{x^m}{\prod_{j=1}^{m} (1 - j\, x)}\,. \tag{5}$$

□

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Note 9. The *Stirling* triangle of the second kind $\mathbf{S}$ is an example of an exponential (*aka* binomial) convolution triangle. See *Knuth*'s paper [3] or *Roman*'s book [4]. Here, however, the *o.g.f.* , and not the exponential generating function (*e.g.f.*) $\frac{1}{m!}(\exp(x) - 1)^m$, $m \geq 0$, is of interest (these types of generating functions are related by a *Laplace* transform).

Lemma 10. *If sequence D is the BINOMIAL transform of sequence C, i.e., $D = \mathbf{P}\, C$, then their o.g.f. s are related like*

$$\mathcal{D}(x) \;=\; \frac{1}{1-x}\, \mathcal{C}\left(\frac{x}{1-x}\right)\ . \tag{6}$$

This lemma is stated in *Bernstein* and *Sloane* [1] and several references are given there. The proof is elementary. It uses an interchange of the two summations and the recognition of the *o.g.f.* $P_m(x)$ of the mth column of the *Pascal* triangle which has been given in lemma 5.

Corollary 11. *The BINOMIAL transformed sequence of the mth column sequence of the triangle $\mathbf{S}$ of the Stirling numbers of the second kind A048993 has the o.g.f.*

$$PS_m(x) \;=\; \frac{1}{1-x}\, S_m\left(\frac{x}{1-x}\right) \;=\; \frac{1}{x}\, S_{m+1}(x)\,. \tag{7}$$

The last eq. follows from lemma 6.

The proof of proposition 1 is now clear because the BINOMIAL transform of the *Bell* sequence $\{B(n)\}_0^\infty$, *i.e.*, $\mathbf{P}\, B$, is also the row sum of the BINOMIAL transformed matrix $\mathbf{S}$, where each column is transformed. *I.e.*, the *Bell* sequence is the row sum of the matrix product $\mathbf{P}\,\mathbf{S}$. This follows from $PS_m(x) = (\mathbf{P}\,\mathbf{S})_m(x)$, the *o.g.f.* of the mth column sequence of the matrix $\mathbf{P}\,\mathbf{S}$. Thus,

$$\sum_{m=0}^{\infty} PS_m(x) \;=\; \frac{1}{x}\sum_{m=0}^{\infty} S_{m+1}(x) \;=\; \frac{1}{x}\sum_{m=1}^{\infty} S_m(x) \;=\; \frac{1}{x}\left(\mathcal{B}(x) \,-\, 1\right) . \qquad (8)$$

The last step follows from lemmata 6 and 8. This is the *o.g.f.* for the shifted *Bell* sequence $\{B(n+1)\}_{n=0}^\infty = [1, 2, 5, ...]$ with offset 0. □

3 Powers of matrix A071919

Proposition 12.

$$\lim_{n\to\infty} \mathbf{T}^n \;=\; (\vec{B}, \vec{0}, \vec{0}, ...) \;, \qquad (9)$$

with the (infinite dimensional) lower triangular matrix $\mathbf{T}$ = *A071919 (extended Pascal), the (infinite dimensional) vector* $\vec{B}$ *with entries* $(B(0) = 1, B(1) = 1, B(2) = 2, ...)^\top$*, the Bell sequence A000110, and the vector* $\vec{0}$ *with only* 0 *entries. This implies that the Bell sequence* $\vec{B}$ *is the asymptotic vector for the vector* $(1, 0, 0, 0, ...)^\top$ *under* $\mathbf{T}$ *iteration.*

Note 13. That the first column becomes the *Bell* sequence stems from its property α stated in *Bernstein* and *Sloane* [1].

Proof. Several lemmata are formulated first.

Note 14. In this section the interest is on the matrix $\mathbf{T}$ =A071919. See the definition 3 and compare with the BINOMIAL transform 2 which uses the *Pascal* matrix $\mathbf{P}$ = A007318 in place of $\mathbf{T}$.

Lemma 15. *The o.g.f. of the* m*th column sequence of the matrix* $\mathbf{T}$ = *A071919 is for* $m = 0$ *given by* $T(0; x) = T(x) = \dfrac{1}{1-x}$ *and for* $m \geq 1$ *by* $T(m; x) = (x\, T(x))^{m+1}$*.*

This is obvious from the *o.g.f.* s of the shifted *Pascal* triangle columns. Note the difference in notation between $T(n, m)$ and $T(m; x)$.

Note 16. $\mathbf{T}$ is, in the strict sense, not a *Riordan triangle* because the columns are not obtained *via* convolution. This means that the *o.g.f.* of the mth column is not of the type $\mathcal{G}(x)(x\,\mathcal{F}(x))^m$ with some $\mathcal{G}$ and $\mathcal{F}$ with $\mathcal{F}(0) = 1$. Of course, the given column *o.g.f. s are also simple to manage.*

Lemma 17. *The o.g.f.* $\mathcal{D}(x) = \sum_{n=0}^{\infty} D(n)\, x^n$ *of the* $\mathbf{T}$ = *A071919-transformed sequence* D *of the sequence* C *with o.g.f.* $\mathcal{C}(x) = \sum_{m=0}^{\infty} C(m)\, x^m$ *is*

$$\mathcal{D}(x) \;=\; \mathcal{C}(0) \;+\; \frac{x}{1-x}\,\mathcal{C}\left(\frac{x}{1-x}\right)\;. \tag{10}$$

Proof. This runs along the lines of the proof of lemma 10.

$$\begin{aligned}
\mathcal{D}(x) \;&=\; \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(n,m)\,C(m)\,x^n \;\;,\;\text{interchanging the summations yields:}\\
&= \sum_{m=0}^{\infty} C(m) \sum_{n=m}^{\infty} T(n,m)\,x^n \;=\; \sum_{m=0}^{\infty} C(m)\,T(m;x)\\
&= C(0)\,T(0;x) \;+\; x\,T(x)\sum_{m=1}^{\infty} C(m)\,(x\,T(x))^m\\
&= \mathcal{C}(0)\,T(x) \;+\; x\,T(x)\,(\mathcal{C}\left(\frac{x}{1-x}\right) \;-\; \mathcal{C}(0)) \;=\; \mathcal{C}(0) \;+\; x\,T(x)\,\mathcal{C}\left(\frac{x}{1-x}\right)\;. \qquad (11)
\end{aligned}$$

□

Corollary 18. *The o.g.f.* $T(n,m;x)$ *of the* m*th column of the* n*th power of the matrix* $\mathbf{T}$ = *A071919 satisfies the recurrence relation*

$$\begin{aligned}
T(n,m;x) \quad &= \quad T(n-1,m;0) \;+\; \frac{x}{1-x}\,T(n-1,m;\frac{x}{1-x})\,,\;\; \textit{for } n\,\geq 1,\; m\,\geq 0,\\
\textit{with input} \qquad\quad T(1,m;x) \;&=\; T(m;x) \;=\; \begin{cases} T(x) \;=\; \frac{1}{1-x}\;, & \textit{if } m=0 \\ (x\,T(x))^{m+1}\,, & \textit{if } m\,\geq 1 \end{cases}\;. \qquad (12)
\end{aligned}$$

This uses $\mathbf{T}^n \;=\; \mathbf{T}\,\mathbf{T}^{n-1}$ and lemma 17 is applied for the mth column sequence of the matrix $\mathbf{T}^{n-1}$ whose *o.g.f.* is $T(n-1,m;x)$.

Lemma 19. *The o.g.f. of the sequence of the first* ($m=0$) *column of the matrix power* $\mathbf{T}^n$ *is*

$$T(n,0;x) \;=\; \frac{1}{\prod_{j=1}^{n-2}(1\;-\;j\,x)}\,\frac{p(n,x)}{1-n\,x}\,,\quad n\,\geq 1. \tag{13}$$

For $n\,=\,1$ *and* 2 *the product has to be replaced by* 1*. The recurrence for the polynomials* $p(n,x)$ *of degree* $n-1$ *is*

$$p(n,x) \;=\; \prod_{j=1}^{n-2}(1\;-\;j\,x)\,(1-n\,x) \;+\; x\,(1-x)^{n-2}\,p\left(n-1,\frac{x}{1-x}\right) \quad n\,\geq 2\,, \tag{14}$$

with input $p(1,x) \;=\; 1$.

Proof. The recurrence from corollary 18 is for $m = 0$

$$T(n, 0; x) \;=\; T(n-1, 0; 0) \;+\; \frac{x}{1-x}\, T\left(n-1, 0; \frac{x}{1-x}\right) \,, \tag{15}$$

with input $T(1, 0; x) = T(x) = \frac{1}{1-x}$. The ansatz for $T(n, 0; x)$ as in lemma 19 leads to the recurrence stated for the polynomials p. Note that $p(n, 0) = 1$ for all $n \geq 1$. It follows that they are integer polynomials of the given degree. Now the lemma is proved with this recurrence by induction, and this proof is left to the reader. □

Corollary 20. *The integer coefficients of the polynomial system p from lemma 19 constitute the triangular array A157165 which starts like* $[[1], [1, -1], [1, -3, 1], [1, -6, 9, -3],$ $[1, -10, 32, -37, 11], [1, -15, 81, -192, 189, -53], ...]$.

Lemma 21. *$T(n, 0; x)$ from lemma 19 can be written as*

$$T(n, 0; x) \;=\; 1 \;+\; \sum_{k=1}^{n} \frac{x^k}{\prod_{j=1}^{k} (1 - j\, x)} \,, \tag{16}$$

which is the nth partial sum of the o.g.f. of the Bell sequence known from lemma 8.

Proof. $T(n, 0; x)$ of the this lemma satisfies the recurrence given in the proof of lemma 19 with the correct input $T(1, 0; x) = 1 + \frac{x}{1-x} = \frac{1}{1-x} = T(x)$. Then mathematical induction over n is used. For $n = 1$ the statement is correct due to the correct input. Now one assumes that the formula is true for all $p = 1, 2, ..., n$. From the recurrence (written for $n \to n+1$), using the induction hypothesis, one obtains

$$\begin{aligned} T(n+1, 0; x) \;&=\; T(n, 0; 0) \;+\; \frac{x}{1-x} \sum_{k=0}^{n} \frac{(x/(1-x))^k}{\prod_{j=1}^{k} (1 - \; j\, x/(1-x))} \\ &= 1 \;+\; \frac{x}{1-x} \sum_{k=0}^{n} \frac{x^k}{\prod_{j=1}^{k} (1 - \; (j+1)\, x)} \;=\; 1 \;+\; \frac{x}{1-x} \sum_{k=0}^{n} \frac{x^k}{\prod_{j=2}^{k+1} (1 - \; j\, x)} \\ &= 1 \;+\; \sum_{k=0}^{n} \frac{x^{k+1}}{\prod_{j=1}^{k+1} (1 - \; j\, x)} \;=\; 1 \;+\; \sum_{k=1}^{n+1} \frac{x^k}{\prod_{j=1}^{k} (1 - \; j\, x)} \,, \end{aligned} \tag{17}$$

which is indeed the claimed formula for $T(n+1, 0; x)$. □

Lemma 22. *$T(n, m; x)$, the o.g.f. of the mth column sequence of $\mathbf{T}^n$, is for $m \geq 1$ given by*

$$T(n, m; x) \;=\; x^{n+m} \frac{1}{\prod_{j=1}^{n-1} (1 - \; j\, x)} \, \frac{1}{(1 - \; n\, x)^{m+1}} \,. \tag{18}$$

Proof. The recurrence from corollary 18 with input $T(1, m; x) = T(m; x) = (x\, T(x))^{m+1}$ is employed. The proof runs with induction over n, for fixed $m \geq 1$, and is left to the reader. □

Now the proof of proposition 12 is clear. For $m = 0$ one finds from lemma 19, in the limit $n \to \infty$, the (formal) *o.g.f.* $\mathcal{B}(x)$ for the *Bell* sequence, known from lemma 8. For each $m \geq 1$ one obtains in the limit $n \to \infty$, because of the pre-factor x^{n+m} in eq. 18, the $\vec{0}$ sequence. (The number of leading 0 members of the sequence grows with n).

□

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