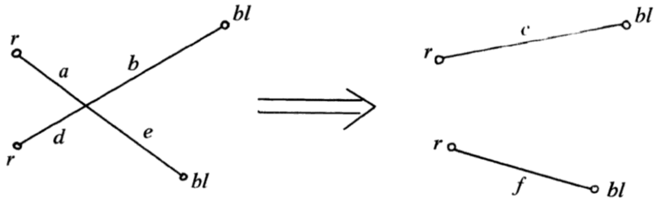
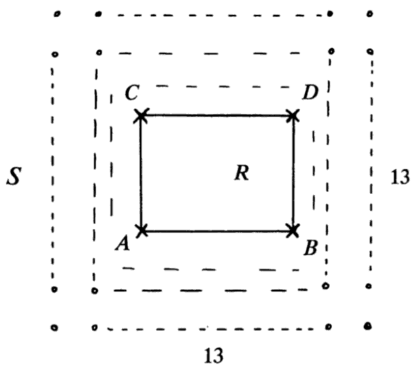


undo the intersection as shown, keeping the other $n - 2$ segments unchanged ($a + b > c, d + e > f$). Consequently, any arrangement which has minimum total length must necessarily be free of intersections.

2. Two Problems from the 1974 USSR National Olympiad (#4 and #9)

> #4. Consider a square grid S of 169 points which are uniformly arrayed in 13 rows and 13 columns (like the lattice points (m, n), $m, n = 1, 2, \ldots, 13$). Prove that no matter what subset T, consisting of 53 of these points, might be selected, some 4 points of T will be the vertices of a rectangle R whose sides are parallel to the sides of S.



We need to show that some pair of points (A, B) of T, in one row of S, line up in the same pair of columns with a second such pair (C, D) in another row. Suppose the rows of S are numbered $1, 2, \ldots, 13$ and that the number of points of T in row i is a_i.

Now a_i points in the same row determine $\binom{a_i}{2}$ pairs of candidates (A, B), each occurring in one of the $\binom{13}{2}$ possible pairs of columns of S. Of course, there is a very good chance that none of the $\binom{a_i}{2}$ pairs of columns determined by the points of T in a particular row i will occur again among the $\binom{a_j}{2}$ such pairs for any other row j. But, if the *total* number of pairs of columns determined by the rows of points of T, namely $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_{13}}{2}$, were to exceed $\binom{13}{2}$, the number of possible pairs of columns of S, the pigeonhole principle* would imply that some pair of columns would have to be repeated, and thus produce a desired rectangle R. Therefore, let us try to show that

$$\sum_{i=1}^{13} \binom{a_i}{2} > \binom{13}{2},$$

which simplifies easily as follows (since T contains 53 points, we have $\sum_{i=1}^{13} a_i = 53$):

$$\sum_{i=1}^{13} \frac{a_i(a_i - 1)}{2} > \frac{13 \cdot 12}{2},$$

$$\sum_{i=1}^{13} (a_i^2 - a_i) > 13 \cdot 12,$$

$$\sum_{i=1}^{13} a_i^2 > 156 + \sum_{i=1}^{13} a_i = 156 + 53 = 209.$$

By setting each $b_i = 1$ in the famous Cauchy inequality*

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

(which, I expect, would be well known to all the olympiad mathletes, who are thoroughly coached these days), we obtain

$$\left(\sum_{i=1}^{13} a_i^2\right) \cdot 13 \geq \left(\sum_{i=1}^{13} a_i\right)^2,$$

* An asterisk indicates a word or idea that is explained in the glossary.

which gives

$$\sum_{i=1}^{13} a_i^2 \geq \frac{53^2}{13} > 53 \cdot \frac{52}{13} = 53 \cdot 4 = 212 > 209,$$

as desired.

This companion problem approaches the same subject from the opposite point of view, a nice touch by the composers of this olympiad.

> #9. Given a square grid S containing 49 points in 7 rows and 7 columns, a subset T consisting of k points is selected. The problem is to find the maximum value of k such that no 4 points of T determine a rectangle R having sides parallel to the sides of S.

Using the notation established in the previous problem, we see immediately that unless

$$\sum_{i=1}^{7} \binom{a_i}{2} \leq \binom{7}{2},$$

an undesired rectangle R will surely result. Since $\Sigma_{i=1}^{7} a_i = k$, this reduces to

$$\sum_{i=1}^{7} a_i^2 \leq 42 + k \tag{1}$$

Turning again to the Cauchy inequality, putting each $b_i = 1$ yields

$$(a_1 + a_2 + \cdots + a_7)^2 \leq \left(\sum_{i=1}^{7} a_i^2\right) \cdot 7,$$

and

$$\frac{k^2}{7} \leq \sum_{i=1}^{7} a_i^2 \tag{2}$$

Combining (1) and (2), we obtain

$$\frac{k^2}{7} \leq \sum_{i=1}^{7} a_i^2 \leq 42 + k,$$

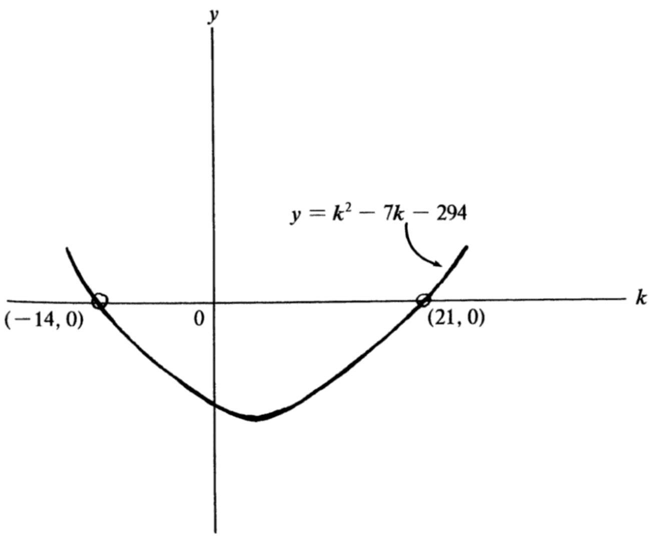
which demands that

$$\frac{k^2}{7} \leq 42 + k,$$

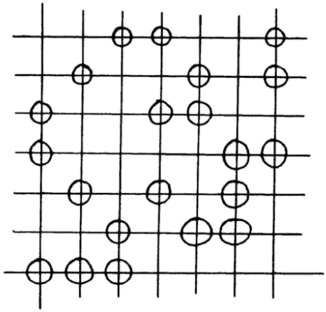
$$k^2 - 7k - 294 \leq 0,$$

$$(k + 14)(k - 21) \leq 0,$$

placing k in the range $-14 \leq k \leq 21$.



Is it possible for k to be as large as 21? If $k = 21$, all of the inequalities, including the Cauchy inequality, become equalities. Now there is equality in Cauchy's relation only if the a_i and b_i are, respectively, proportional. Since all $b_i = 1$, equality here is out of the question unless the a_i's are all equal. Thus, if k can actually be as great as 21, T will have to contain exactly 3 points from each row. In checking the feasibility of such an arrangement by direct trial, one soon succeeds as shown. Thus the maximum k is indeed 21.



3. Stanley's Theorem

Every now and again one comes across an astounding result that closely relates two foreign objects which seem to have nothing in common. Who would suspect, for example, that, on the average, the number of ways of expressing a positive integer n as a sum of two integral squares, $x^2 + y^2 = n$, is π ([**3**]). In this section I would like to tell you about another of these totally unexpected results, a delightful little gem due to Richard Stanley of MIT.

The 11 unordered partitions of the positive integer 6 are listed in Table 1 below. A second column in the table gives the number of distinct parts (i.e., repetitions are not counted) that occur in the partitions. If we add up the numbers in this second column we obtain a total of 19. Now count the number of 1's that occur in the list of partitions. Hmmm . . . isn't that interesting? This is no coincidence, for the same is true for the partitions of every positive integer.

STANLEY'S THEOREM. *The total number of* 1*'s that occur among all unordered partitions of a positive integer is equal to the sum of the numbers of distinct parts of those partitions.*

Let us denote the number of unordered partitions of n by $p(n)$, and define $p(0)$ to be 1. Since the order of the integers in a partition doesn't count, rearranging them to suit ourselves doesn't cause any trouble. Consequently, let us write them in nondecreasing order and enter them in a table (in a normal way—one partition per row, starting each at the left). Each partition will occupy as many columns as it