Anti-Fibonacci Numbers

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The recursive rule defining the “anti-Fibonacci” numbers is:

The next member of the anti-Fibonacci sequence is the sum of the two most recent NON-members of the anti-Fibonacci sequence.

We further stipulate that the first member of the sequence is 0.

The first two non-members being 1 and 2, their sum is 3, so that’s the second member of the sequence. The next two non-members are 4 and 5, ergo the third member of the sequence 3 is 9. The next two non-members are 6 and 7, so the fourth member is 13. The next two non-members are 8 and 10, so the fifth member is 18. (Etc. etc.)

Below is a picture showing the sequence of members (green) and that of non-members (red); together, by definition, these two sequences make up the whole whole-number line.

```
    MEMBERS:  0      3               9            13                 18

NON-MEMBERS:     1  2   4  5  6  7  8   10  11  12    14  15  16  17    19  20
```

Below are displayed the first 14 anti-Fibonacci numbers, together with their first differences:

```
0   3   9   13   18   23   29   33   39   43   49   53   58   63...

  3   6   4    5    5    6    4    6    4    6    4    5    5
```

The rule for the first-difference sequence is that, aside from the initial "3", one always has either "6-4" (i.e., a 6 followed by a 4), or else "5-5". The 6-4's always come either alone (6-4) or triply (6-4, 6-4, 6-4), while the 5-5's always come alone. So if we let "6-4, 5-5" be represented by the numeral "1", and "6-4, 6-4, 6-4, 5-5" by the numeral "3", then the sequence of first differences, in this compressed code, comes down to this:

1311333113113113331113...

This sequence can be captured in a multi-level pyramidal tree as follows:

```
          1
          3
         113
        33113
     11311333113
 33113331131131133311 3
```

Two simple rules show how a a new line is derived from the line above it:

1 ==> 3   3 ==> 113

(These rules are in fact illustrated by the top three lines of the tree.) That’s all!

Notice that 3, 13, 23, 33, 43, 53, 63, 73, 83, ... (numbers having the form “X3” in decimal notation) are always anti-Fibonacci numbers, and wedged between them always comes either X8 or X9 (thus 9, 18, 29, 39, 49, 58, ...). Each X9 corresponds to a 6-4, and each X8 corresponds to a 5-5, so we could alternatively encode the anti-Fibonacci sequence by writing “98 9998 98 98 9998 9998 9998 98 98...”. If we were then to replace “98” by “1” and 9998” by “3”, we would once again get the sequence 1311333113113113331113...