Anti-Fibonacci Numbers: A Formula

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The non-anti-Fibonacci numbers (see [2, Sequence A249031]) are the unparenthesised numbers in

$$\begin{aligned} &1,\ 2,\ (3=1+2),\ 4,5,6,7,8,(9=4+5),\ 10,\ 11,\ 12,\\ &\quad(13=6+7),\ 14,\ 15,\ 16,\ 17,\ (18=8+10),\ 19,\ 20,\ 21,\ 22,\\ &\quad(23=11+12),\ 24,\ \ldots; \end{aligned}$$

this is [2, Sequence A249031]. The numbers in parentheses are the *anti-Fibonacci numbers* $\bar{F}_n$ (with $\bar{F}_0 = 3$); they are defined as the sums of pairs of consecutive non-anti-Fibonacci natural numbers (beginning with 1), each non-anti-Fibonacci number occurring in one such sum. These anti-Fibonacci numbers are

$$3,\ 9,\ 13,\ 18,\ 23,\ 29,\ 33,\ 39,\ \ldots;$$

they are [2, Sequence A075326] (we omit 0, which does not fit the pattern). We deduce an anti-Fibonacci number formula from the jumps (first-order differences, [2, Sequence A249032]) in the sequence.[1]

Define an integer to be *utterly odd* if the terminal string of 1's in its binary representation has odd length. For instance, that string has the even length 0 for an even integer. A number $2^{k+1}m + (2^k - 1)$ where $m \geq 0$ (every non-negative integer has this form) is utterly odd if and only if k is odd. Utterly odd positive numbers are

$$1, 5, 7, 9, 13, 17, 21, 23, 25, 29, 31, \ldots.$$

The *utterly odd nature* of an integer is the property of being, or not being, utterly odd. Most odd integers are utterly odd; those that are not are

$$3, 11, 15, 19, 27, 35, \ldots$$

(see [2, Sequence A131323]).

Theorem 1. *The anti-Fibonacci numbers $\bar{F}_n$ of the second kind, indexed so $\bar{F}_0 = 3$, are*

$$\bar{F}_n = \begin{cases} 5n+3 & \text{if } n \text{ is even,} \\ 5n+3 & \text{if } n \text{ is odd and } (n-1)/2 \text{ is not utterly odd,} \\ 5n+4 & \text{if } n \text{ is odd and } (n-1)/2 \text{ is utterly odd.} \end{cases}$$

Proof. This is a restatement of Theorem 2. □

Theorem 2. *Indexing so that $\bar{f}_0 = 4$, the non-anti-Fibonacci numbers are*

$$\bar{f}_n = \frac{1}{4}[5n - (n \mod 8)] + c \qquad \text{for } n \geq -2,$$

[1] I am grateful to Tao-Ming Wang for leading me deep into this question at Indira Gandhi International Airport.

where

$$c = \begin{cases} 4 & \textit{if } (n \mod 8) < 4, \textit{ or if } (n \mod 8) = 4 \textit{ and } \lfloor n/8 \rfloor \textit{ is utterly odd,} \\ 5 & \textit{if } (n \mod 8) > 4, \textit{ or if } (n \mod 8) = 4 \textit{ and } \lfloor n/8 \rfloor \textit{ is not utterly odd.} \end{cases}$$

By $(n \mod m)$ I mean the least non-negative residue of n modulo m, which is a number in the range $0, 1, \ldots, m-1$.

The difficult part was finding the formula, which I did by interpreting the following replication lemma as defining the locations and relationships of missing numbers.

Lemma 1. *The anti-Fibonacci number sequence* $3, 9, 13, 18, 23, \ldots$ *has jumps that occur in consecutive pairs* $A = (6, 4)$ *or* $B = (5, 5)$*. The pattern of jump pairs is generated from the sequence* A *by the substitution rules* $A \mapsto AB$ *and* $B \mapsto AA$*.*

Proof. The first four jumps, which are AB, follow this rule. The rest of the proof proceeds by induction after some preparation.

Consider four consecutive non-anti-Fibonacci numbers, $\bar{f}_i, \bar{f}_{i+1}, \bar{f}_{i+2}, \bar{f}_{i+3}$ (call them, collectively, C), that constitute two summed pairs. The pairs sums $s_1 = \bar{f}_i + \bar{f}_{i+1}$ and $s_2 = \bar{f}_{i+2} + \bar{f}_{i+3}$ satisfy $s_2 > s_1 + 3$. Therefore, any two anti-Fibonacci numbers differ by at least 4. Furthermore, they differ by 4 only if C is four consecutive integers; otherwise they differ by 5. There is no other possibility because anti-Fibonacci numbers differ by at least 4.

Assume, as is true initially, that eight consecutive non-anti-Fibonacci numbers, call them E, are $\bar{f}_{8a} = 10a+4, \ldots, \bar{f}_{8a+7} = 10a+12$ with one anti-Fibonacci number internally at $10a+8$ or $10a+9$, they are preceded by the anti-Fibonacci number $10a+3$, and the summed pairs in E begin with $\bar{f}_{8a} = 10a+4$ and $\bar{f}_{8a+1} = 10a+5$. This sequence generates anti-Fibonacci numbers $\bar{f}_{8a} + \bar{f}_{8a+1} = 20a+9$, $\bar{f}_{8a+2} + \bar{f}_{8a+3} = 20a+13$, $\bar{f}_{8a+4} + \bar{f}_{8a+5} = 20a+18$ or $20a+19$, and $\bar{f}_{8a+6} + \bar{f}_{8a+7} = 20a+23$. The jumps generated by E are 6 from $\bar{f}_{8a-2} + \bar{f}_{8a-1} = 20a+3$, 4, 5 or 6, and 5 or 4; thus, they are 6, 4, 5, 5 or 6, 4, 6, 4, also known as AB or AA. Thus, the pattern of jumps A or B in a decade of integers $8a+3, \ldots, 8a+12$ replicates itself (imperfectly) in the two decades $16a+3, \ldots, 16a+22$ as AB or AA, respectively. That proves the first and second assertions of the lemma. □

Proof of Theorem 2. We already established in the course of proving Lemma 1 that the non-anti-Fibonacci numbers have the values in Theorem 2, except that we have not determined when c in the one ambiguous residue class $n \equiv 4 \mod 8$ equals 4 or 5. The Theorem does give the right values for $\bar{f}_0, \ldots, \bar{f}_{15}$, so we can perform an induction.

Let's perform a few steps of replication. We get A, B sequences

Step 0: A
Step 1: AB
Step 2: $ABAA$
Step 3: $ABAA\ ABAB$
Step 4: $ABAA\ ABAB \quad ABAA\ ABAA$

in which the locations of B's are at the following positions, beginning at position 0 and written in binary. Parentheses denote positions with an A. We omit even numbers because all even positions are occupied by A's.

Step 0: –

Step 1: 1

Step 2: 01 (11)

Step 3: 001 (011) 101 111

Step 4: 0001 (0011) 0101 0111   1001 (1011) 1101 (1111)

We call a sequence of consecutive odd numbers from 0 to $2^k - 1$ in fixed-length binary, ignoring parentheses, a *binary odd sequence* and the step from one to the next *doubling*. We observe that the B's are in the utterly odd positions. We also see that the first and second halves of each sequence are identical except for the very last element, which differs, alternating between A in the first half and B in the second, and the reverse.

Now we show that pattern continues. First, we show that replication preserves that pattern in A, B strings. Let ρ denote the replication operator. If we have an A, B pattern Π of the form $\Sigma\alpha\Sigma\bar{\alpha}$ where Σ is a string of A's and B's, α denotes either A or B, and $\bar{\alpha}$ is the opposite letter, then

$$\rho(\Pi) = \rho(\Sigma)\rho(\alpha)\rho(\Sigma)\rho(\bar{\alpha}) = \rho(\Sigma)A\bar{\alpha}\rho(\Sigma)A\alpha.$$

Thus, the result has the form $\Sigma'\bar{\alpha}\ \Sigma'\alpha$, the same shape as Π but with the terminal letter of each half reversed.

Now we show that doubling a binary odd sequence transforms it in the same way. Write δ for the doubling operator. Consider a binary odd sequence $\pi = (\sigma_0, \ldots, \sigma_{2^k-1})$ where $\sigma_j = \beta_{j,k-1} \cdots \beta_{j,1}1$ is a string of length k and each $\beta_{j,i} \in \{0, 1\}$. In terms of binary strings, δ transforms π to

$$\delta(\pi) = (0\sigma_0, \ldots, 0\sigma_{2^k-1})(1\sigma_0, \ldots, 1\sigma_{2^k-1}),$$

where juxtaposition of sequences denotes concatenation. The string $0\sigma_j$ has the same utterly odd nature as does σ_j. The string $1\sigma_j$ has the same utterly odd nature as σ_j does if there is a 0 in σ_j. The only way $1\sigma_j$ can differ from σ_j is for σ_j to consist entirely of 1's; then $1\sigma_j$ has the opposite nature to σ_j. This proves that, if at some Step k in the application of ρ and δ the B's appear exactly where there are utterly odd numbers, then the same holds true at Step $k + 1$. The theorem is therefore proved. □

Hofstadter [1] used rewriting rules to develop a recursive construction for the sequence $\{\bar{F}_n\}$, but his rules are not doubling rules. Possibly for that reason, he did not detect the binary rule for locating B's that led me to our theorems.

References

[1] Doug Hofstadter, Anti-Fibonacci numbers. Manuscript, 2014. `http://oeis.org/A075326/a075326_1.pdf`

[2] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences.* `http://oeis.org/`