# Proofs of various conjectures about the sequences A075827, A075828, A075829, and A075830

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Define the sequence $(u(n,x) : n \geq 1)$ of rational functions of $x$ by

$$u(1,x) = x \quad \text{and} \quad u(n+1,x) = \frac{n^2}{u(n,x)} + 1 \quad \text{for } n \geq 1.$$

In this note, we prove various conjectures about the above rational sequence related to the OEIS sequences A075827, A075828, A075829, and A075830. These sequences were originally defined by Benoit Cloitre in 2002. Let

$$\alpha(n) = \underline{A024167}(n) = n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \quad \text{and} \quad \beta(n) = \underline{A024168}(n) = n! \sum_{k=2}^{n} \frac{(-1)^{k}}{k}. \tag{1}$$

**Theorem 1.** *For each integer $n \geq 2$, we have*

$$u(n,x) = \frac{n\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right)x + n\left(\sum_{k=2}^{n} \frac{(-1)^{k}}{k}\right)}{\left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}\right)x + \left(\sum_{k=2}^{n-1} \frac{(-1)^{k}}{k}\right)} \tag{2}$$

$$= \frac{\underline{A024167}(n)\,x + \underline{A024168}(n)}{\underline{A024167}(n-1)\,x + \underline{A024168}(n-1)}. \tag{3}$$

*Proof.* Equation (3) follows from equation (2) by multiplying the numerator and denominator of the fraction in (2) by $(n-1)!$.

We prove equation (2) by induction on $n$. For $n = 2$, we have

$$u(2,x) = \frac{1^2}{u(1,x)} + 1 = \frac{x+1}{x} = \frac{2\left(1 - \frac{1}{2}\right)x + 2\left(\frac{1}{2}\right)}{(1)x + 0},$$

and the base case for induction has been established.

Next we proceed with the induction step. Assume equation (2) holds for an arbitrary $n \geq 2$. Then

$$u(n+1,x) = \frac{n^2}{u(n,x)} + 1 = \frac{n\left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}\right)x + n\left(\sum_{k=2}^{n-1} \frac{(-1)^{k}}{k}\right)}{\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right)x + \left(\sum_{k=2}^{n} \frac{(-1)^{k}}{k}\right)} + 1$$

$$= \frac{n\left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}\right)x + n\left(\sum_{k=2}^{n-1} \frac{(-1)^{k}}{k}\right) + \left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right)x + \left(\sum_{k=2}^{n} \frac{(-1)^{k}}{k}\right)}{\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right)x + \left(\sum_{k=2}^{n} \frac{(-1)^{k}}{k}\right)}$$

$$= \frac{(n+1)\left(\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k}\right)x + (n+1)\left(\sum_{k=2}^{n+1} \frac{(-1)^{k}}{k}\right)}{\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right)x + \left(\sum_{k=2}^{n} \frac{(-1)^{k}}{k}\right)}.$$

Thus, equation (2) holds for $n+1$ as well, and this completes the inductive step. $\square$

**Lemma 2.** *For each integer $n \geq 2$, we have*

$$\gcd(\alpha(n), \alpha(n-1)) = \gcd(\beta(n), \beta(n-1)) \tag{4}$$
$$= \gcd(\alpha(n-1), (n-1)!) \tag{5}$$
$$= \gcd(\beta(n-1), (n-1)!) = \gcd(\alpha(n-1), \beta(n-1)), \tag{6}$$

*where the sequences $\alpha$ and $\beta$ are defined in (1).*

*Proof.* Fix integer $n \geq 2$. It is trivial to establish the following identities:

$$\alpha(n) - n\alpha(n-1) = (n-1)!(-1)^{n+1} = -\beta(n) + n\beta(n-1), \tag{7}$$

$$\alpha(n-1) + \beta(n-1) = (n-1)! \quad \text{and} \quad \alpha(n) + \beta(n) = n!. \tag{8}$$

Let $\alpha^* = \gcd(\alpha(n), \alpha(n-1))$ and $\beta^* = \gcd(\beta(n), \beta(n-1))$. Identities (7) imply that $\alpha^*|(n-1)!$ and $\beta^*|(n-1)!$. Using these and identities (8), we get the following:

$$(\alpha^*|\alpha(n-1) \ \& \ \alpha^*|(n-1)!) \Longrightarrow \alpha^*|\beta(n-1) \quad \text{and} \quad (\alpha^*|\alpha(n) \ \& \ \alpha^*|(n-1)!) \Longrightarrow \alpha^*|\beta(n).$$

It follows that $\alpha^*|\beta^*$. In a similar way, we can prove that $\beta^*|\alpha^*$. It follows that $\alpha^* = \beta^*$.

Now let $\alpha^{**} = \gcd(\alpha(n-1), (n-1)!)$ and $\beta^{**} = \gcd(\beta(n-1), (n-1)!)$. From equations (7), we get that $\alpha^{**}|\alpha(n)$ and $\beta^{**}|\beta(n)$. But trivially we have $\alpha^{**}|\alpha(n-1)$ and $\beta^{**}|\beta(n-1)$. Thus,

$$\alpha^{**}|\gcd(\alpha(n), \alpha(n-1)) = \alpha^* \quad \text{and} \quad \beta^{**}|\gcd(\beta(n), \beta(n-1)) = \beta^*.$$

But from equations (7), we also get $\alpha^*|(n-1)!$ and $\beta^*|(n-1)!$. But we trivially have $\alpha^*|\alpha(n-1)$ and $\beta^*|\beta(n-1)$. Hence

$$\alpha^*|\gcd((n-1)!, \alpha(n-1)) = \alpha^{**} \quad \text{and} \quad \beta^*|\gcd((n-1)!, \beta(n-1)) = \beta^{**}.$$

Combining all of the above results, we conclude that $\alpha^* = \alpha^{**} = \beta^{**} = \beta^*$.

Finally, let $\gamma^* = \gcd(\alpha(n-1), \beta(n-1))$. From the first equation in (8), we get $\gamma^*|(n-1)!$. Since also $\gamma^*|\alpha(n-1)$, we conclude that $\gamma^*|\gcd(\alpha(n-1), (n-1)!) = \alpha^{**}$. From the first equation in (8), we also have $\alpha^{**}|\beta(n-1)$. Since $\alpha^{**}|\alpha(n-1)$, we get $\alpha^{**}|\gcd(\beta(n-1), \alpha(n-1)) = \gamma^*$. Thus, $\gamma^* = \alpha^{**}$, and this finishes the proof of the lemma. □

We let $\gamma(n)$ denote the sequence described in equations (4), (5), and (6) of Lemma 2; that is,

$$\begin{aligned} \gamma(n) = \underline{A334958}(n-1) &= \gcd(\alpha(n), \alpha(n-1)) = \gcd(\beta(n), \beta(n-1)) \\ &= \gcd(\alpha(n-1), (n-1)!) = \gcd(\beta(n-1), (n-1)!) \\ &= \gcd(\alpha(n-1), \beta(n-1)) \quad \text{for } n \geq 2. \end{aligned}$$

Define now the sequences $v_1, v_2, v_3, v_4$ as follows:

$$v_1(1) = 1 \quad \text{and} \quad v_1(n) = \frac{\alpha(n)}{\gamma(n)} \quad \text{for } n \geq 2;$$

$$v_2(1) = 0 \quad \text{and} \quad v_2(n) = \frac{\beta(n)}{\gamma(n)} \quad \text{for } n \geq 2;$$

$$v_3(1) = 0 \quad \text{and} \quad v_3(n) = \frac{\alpha(n-1)}{\gamma(n)} \quad \text{for } n \geq 2;$$

$$v_4(1) = 1 \quad \text{and} \quad v_4(n) = \frac{\beta(n-1)}{\gamma(n)} \quad \text{for } n \geq 2.$$

We shall prove that

$$v_1 = \underline{A075827}, \quad v_2 = \underline{A075828}, \quad v_3 = \underline{A075830}, \quad v_4 = \underline{A075829}.$$

This follows immediately from the following theorem.

**Theorem 3.** *For all integer $n \geq 1$,*

$$u(n,x) = \frac{v_1(n)x + v_2(n)}{v_3(n)x + v_4(n)}. \tag{9}$$

*Also, $v_1(n) + v_2(n) = n(v_3(n) + v_4(n))$ and $\gcd(v_3(n), v_4(n)) = 1$ for $n \geq 1$. This means that the rational function above is in lowest terms.*

*Proof.* Equation (9) is obvious for $n = 1$. Assume $n \geq 2$. From equations (1) and (3), we have

$$u(n,x) = \frac{\alpha(n)x + \beta(n)}{\alpha(n-1)x + \beta(n-1)} = \frac{\frac{\alpha(n)}{\gamma(n)}x + \frac{\beta(n)}{\gamma(n)}}{\frac{\alpha(n-1)}{\gamma(n)}x + \frac{\beta(n-1)}{\gamma(n)}} = \frac{v_1(n)x + v_2(n)}{v_3(n)x + v_4(n)}.$$

This proves equation (9) when $n \geq 2$.

For $n = 1$, we have $v_1(n) + v_2(n) = 1 + 0 = 1 = 1(0 + 1) = n(v_3(n) + v_4(n))$. Assume now $n \geq 2$. From equations (8), we get

$$v_1(n) + v_2(n) = \frac{\alpha(n) + \beta(n)}{\gamma(n)} = \frac{n!}{\gamma(n)} = \frac{n(\alpha(n-1) + \beta(n-1))}{\gamma(n)} = n(v_3(n) + v_4(n)).$$

Since $\gamma(n) = \gcd(\alpha(n-1), \beta(n-1))$, we get

$$\gcd(v_3(n), v_4(n)) = \gcd\left(\frac{\alpha(n-1)}{\gamma(n)}, \frac{\beta(n-1)}{\gamma(n)}\right) = 1.$$

This finishes the proof of the theorem. □

Next we give some properties of the sequence $(\gamma(n) : n \geq 2)$.

**Lemma 4.** *For integer $n \geq 2$, $\gamma(n) | \gamma(n+1)$.*

*Proof.* From the equation $\alpha(n) - n\alpha(n-1) = (n-1)!(-1)^{n+1}$ we get that

$$\gamma(n) = \gcd(a(n-1), (n-1)!) | a(n).$$

But trivially we have $\gamma(n) | n!$, so $\gamma(n) | \gcd(a(n), n!) = \gamma(n+1)$. □

**Lemma 5.** *For each integer $n \geq 2$, if $n$ is prime, then $\gamma(n) = \gamma(n+1)$.*

*Proof.* Assume $n$ is prime. By Lemma 4, $\gamma(n) | \gamma(n+1)$.

Assume now $n | a(n)$. From the equation $\alpha(n) - n\alpha(n-1) = (n-1)!(-1)^{n+1}$ we get that $n | (n-1)!$, a contradiction. Thus, $\gcd(a(n), n) = 1$. This together with $\gamma(n+1) = \gcd(\alpha(n), n!)$ imply that $\gamma(n+1) | (n-1)!$. But then $\gamma(n+1) | n\alpha(n-1)$, which implies $\gamma(n+1) | \alpha(n-1)$ (since is $n$ is prime with no common factor with $\gamma(n+1)$). Thus

$$\gamma(n+1) | \gcd((n-1)!, \alpha(n-1)) = \gamma(n).$$

Hence, $\gamma(n) = \gamma(n+1)$. □

We believe the converse of Lemma 5 is also true, but we have not been able to prove it.

**Conjecture 6.** *For each integer $n \geq 2$, if $\gamma(n) = \gamma(n+1)$, then $n$ is prime.*

A consequence of Lemma 5 and Conjecture 6 is the following result.

**Corollary 7.** *If $n = 1$ or $n$ is prime, then $v_1(n) = v_3(n+1)$, i.e., A075827$(n)$ = A075830$(n+1)$. If Conjecture 6 is true, then the converse is also true.*

*Proof.* If $n = 1$, then $v_1(1) = 1 = v_3(2)$. If $n$ is prime, then by Lemma 5 we have $\gamma(n) = \gamma(n+1)$. Thus,

$$v_1(n) = \frac{\alpha(n)}{\gamma(n)} = \frac{\alpha(n)}{\gamma(n+1)} = v_3(n+1).$$

Assume now Conjecture 6 is true and $v_1(n) = v_3(n+1)$. If $n > 1$, then $\frac{\alpha(n)}{\gamma(n)} = \frac{\alpha(n)}{\gamma(n+1)}$, and so $\gamma(n) = \gamma(n+1)$. By Conjecture 6, $n$ is prime. $\square$

Next we prove the claims made by N. J. A. Sloane and Alexander Adamchuk in 2006 about the sequences $v_3$ = A075830 and $v_4$ = A075829, respectively.

**Theorem 8.** *Let $H^*(n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ be the alternating harmonic number. Then, for $n \geq 2$,*

$$v_3(n) = \textit{numerator}(H^*(n-1)) = \text{A058313}(n-1) \quad \textit{and}$$
$$v_4(n) = \textit{denominator}(H^*(n-1)) - \textit{numerator}(H^*(n-1)) = \text{A058312}(n-1) - \text{A058313}(n-1).$$

*Proof.* For $n \geq 2$, we have

$$\begin{aligned} v_3(n) = \frac{\alpha(n-1)}{\gamma(n)} &= \frac{\alpha(n-1)}{\gcd(\alpha(n-1), (n-1)!)} \\ &= \text{numerator}\left(\frac{\alpha(n-1)}{(n-1)!}\right) \\ &= \text{numerator}\left(\frac{(n-1)!H^*(n-1)}{(n-1)!}\right) = \text{numerator}(H^*(n-1)). \end{aligned}$$

Similarly, for $n \geq 2$,

$$\begin{aligned} v_4(n) = \frac{\beta(n-1)}{\gamma(n)} &= \frac{\beta(n-1)}{\gcd(\beta(n-1), (n-1)!)} \\ &= \text{numerator}\left(\frac{\beta(n-1)}{(n-1)!}\right) \\ &= \text{numerator}\left(\sum_{k=2}^{n-1} \frac{(-1)^k}{k}\right) = \text{numerator}\left(1 - H^*(n-1)\right). \end{aligned}$$

But if $H^*(n-1) = \frac{a}{b}$, where $a$ and $b$ are integers with $\gcd(a,b) = 1$ and $b \neq 0$, then $1 - H^*(n-1) = \frac{b-a}{b}$ with $\gcd(b-a, b) = 1$. Thus, $v_4(n) = \text{denominator}(H^*(n-1)) - \text{numerator}(H^*(n-1))$. $\square$

In the spirit of Sloane and Adamchuk's formulas, we now give formulas for sequences $v_1$ and $v_2$.

**Theorem 9.** *Let $H^*(n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ be the alternating harmonic number. Then, for $n \geq 2$,*

$$v_1(n) = \textit{numerator}\left(\frac{nH^*(n)}{H^*(n-1)}\right) \quad \textit{and} \quad v_2(n) = \textit{numerator}\left(\frac{n(1 - H^*(n))}{1 - H^*(n-1)}\right),$$

*where $\infty$ is defined as $\frac{1}{0}$ in lowest terms.*

*Proof.* For integer $n \geq 2$, we have

$$\begin{aligned} v_1(n) &= \frac{\alpha(n)}{\gcd(\alpha(n), \alpha(n-1))} \\ &= \text{numerator}\left(\frac{\alpha(n)}{\alpha(n-1)}\right) \\ &= \text{numerator}\left(\frac{n!H^*(n)}{(n-1)!H^*(n-1)}\right) = \text{numerator}\left(\frac{nH^*(n)}{H^*(n-1)}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} v_2(n) &= \frac{\beta(n)}{\gcd(\beta(n), \beta(n-1))} \\ &= \text{numerator}\left(\frac{\beta(n)}{\beta(n-1)}\right) \\ &= \text{numerator}\left(\frac{n!(1 - H^*(n))}{(n-1)!(1 - H^*(n-1))}\right) = \text{numerator}\left(\frac{n(1 - H^*(n))}{1 - H^*(n-1)}\right). \end{aligned}$$

(For the case $n = 2$, $\beta(1) = 0$ and $\beta(2) = 1$ with $\gcd(\beta(2), \beta(1)) = 1$. In this case, the numerator of $\frac{\beta(2)}{\beta(1)} = \frac{1}{0} = \infty$ is defined to be 1.) This completes the proof of the theorem. □

Define the sequences $(A(n) : n \geq 1)$ and $(B(n) : n \geq 1)$ by

$$A(1) = \infty, \quad A(n+1) = \frac{n^2}{A(n)} + 1 \quad \text{for } n \geq 1 \quad \text{and}$$

$$B(1) = 0, \quad B(n+1) = \frac{n^2}{B(n)} + 1 \quad \text{for } n \geq 1,$$

where $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. (We then have $A(2) = 1$, $B(2) = \infty$, and $B(3) = 1$.)

**Theorem 10.** *For $n \geq 1$,*

$$\begin{aligned} v_1(n) &= \textit{numerator}(A(n)), \quad v_2(n) = \textit{numerator}(B(n)), \\ v_3(n) &= \textit{denominator}(A(n)), \quad v_4(n) = \textit{denominator}(B(n)), \end{aligned}$$

*where $\infty$ in lowest terms is defined as $\frac{1}{0}$.*

*Proof.* Note that, for each $n \geq 1$,

$$A(n) = u(n, \infty) = \lim_{x \to \infty} u(n, x) \quad \text{and} \quad B(n) = u(n, 0).$$

It follows from equation (9) in Theorem 3 that, for each $n \geq 1$,

$$A(n) = \frac{v_1(n)}{v_3(n)} \quad \text{and} \quad B(n) = \frac{v_2(n)}{v_4(n)}. \tag{10}$$

For $n = 1$, we clearly have $\gcd(v_1(1), v_3(1)) = \gcd(1, 0) = 1$ and $\gcd(v_2(1), v_4(1)) = \gcd(0, 1) = 1$. For $n \geq 2$,

$$\gcd(v_1(n), v_3(n)) = \gcd\left(\frac{\alpha(n)}{\gcd(\alpha(n), \alpha(n-1))}, \frac{\alpha(n-1)}{\gcd(\alpha(n), \alpha(n-1))}\right) = 1 \quad \text{and}$$

$$\gcd(v_2(n), v_4(n)) = \gcd\left(\frac{\beta(n)}{\gcd(\beta(n), \beta(n-1))}, \frac{\beta(n-1)}{\gcd(\beta(n), \beta(n-1))}\right) = 1.$$

This means that the fractions in equations (10) are in lower terms. The four equations in the statement of the theorem follow immediately. □

We finally prove Benoit Cloitre's limiting result for the sequence $(u(n, x) : n \geq 1)$.

**Theorem 11.** *For any real number $x \neq 1 - \frac{1}{\log 2}$, we have*

$$\lim_{n\to\infty} |u(n, x) - u(n, 1)| = \lim_{n\to\infty} |u(n, x) - n| = \left|\frac{x-1}{1 + (x-1)\log 2}\right|.$$

*Proof.* It is easy to see that

$$u(n, x) - n = \frac{(-1)^{n+1}(x-1)}{\left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}\right) x + 1 - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}}.$$

Taking absolute values on both sides of the above equality and letting $n \to \infty$, we get the result in the theorem. □