A note on the diagonals of a proper Riordan Array

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We show the exponential generating function (e.g.f.) for the n-th subdiagonal of a proper Riordan Array has the form $\exp(cx)P_n(x)$, where c is a constant and $P_n(x)$ is a polynomial of degree at most n: the polynomials $P_n(x)$ are the e.g.f.'s for the rows of an associated Riordan Array.

The *Riordan Array* associated with a pair of generating functions $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = g_0 + g_1x + g_2x^2 + \cdots$ is the lower triangular array whose k-th column, $k = 0, 1, 2, \ldots$, is generated by $f(x)(xg(x))^k$. The matrix corresponding to the pair of generating functions f, g is denoted by $(f(x), xg(x))$. If we have $g_0 \neq 0$ then the Riordan Array is said to be *proper*; if $g_0 = 0$ then the Riordan Array is said to be *stretched*. In the proper-case the main diagonal is the sequence $\left(1, g_0, g_0^2, g_0^3, \ldots\right)$, which has the e.g.f. $\exp(g_0t)$. We wish to generalize this result to all the subdiagonals of a proper Riordan Array.

Theorem. *With the notation as above, let $R = (f(x), xg(x))$ be a proper Riordan Array. Then the e.g.f. for the n-th subdiagonal of R equals exp$(g_0t) \times$ the e.g.f. for row n of the (possibly stretched) Riordan Array $\widetilde{R}$=$(f(x), g(x) - g_0)$.*

Proof. Let $R_{n,k}$ denote the generic element of the Riordan array R. Then the n-th subdiagonal of R is the sequence $(R_{n,0}, R_{n+1,1}, R_{n+2,2}, ...)$ with e.g.f. $R_{n,0} + R_{n+1,1}t + R_{n+2,2}\frac{t^2}{2!} + \cdots$. From the definition of a Riordan Array, $R_{n,k}$ is the n-th coefficient of the series $f(x)(xg(x))^k$. Using $[x^n]$ to denote the coefficient extractor operator we see that the e.g.f. for the n-th subdiagonal of R is given by

$$\begin{aligned}
& [x^n]f(x) + \left([x^{n+1}]f(x)xg(x)\right)t + \left([x^{n+2}]f(x)(xg(x))^2\right)\frac{t^2}{2!} + \left([x^{n+3}](xg(x))^3\right)\frac{t^3}{3!} + \cdots \\
= \; & [x^n]f(x) + ([x^n]f(x)g(x))\,t + \left([x^n]f(x)g(x)^2\right)\frac{t^2}{2!} + \left([x^n]g(x)^3\right)\frac{t^3}{3!} + \cdots \\
= \; & [x^n]\left\{f(x)\left(1 + g(x)t + g(x)^2\frac{t^2}{2!} + g(x)^3\frac{t^3}{3!} + \cdots\right)\right\} \\
= \; & [x^n]f(x)\exp(tg(x)). \qquad (1)
\end{aligned}$$

Similarly, the e.g.f. for row n of the Riordan array $\widetilde{R}$ is equal to

$$\begin{aligned}
& [x^n]f(x) + [x^n]\left(f(x)(g(x) - g_0\right)t + [x^n]\left(f(x)\left(g(x) - g_0\right)^2\right)\frac{t^2}{2!} + [x^n]\left(f(x)\left(g(x) - g_0\right)^3\right)\frac{t^3}{3!} + \cdots \\
= \; & [x^n]f(x)\exp\left(\left(g(x) - g_0\right)t\right) \\
= \; & \exp\left(-g_0t\right)[x^n]f(x)\exp(tg(x)). \qquad (2)
\end{aligned}$$

The result now follows by comparing (1) and (2). □

In the particular case where R has the form $\left(f(x), \frac{x}{1-x}\right)$ then the array $\widetilde{R} = \left(f(x), \frac{x}{1-x}\right) = R$. So in this case the theorem says the e.g.f. for the n-th subdiagonal of R equals $\exp(t) \times$ the e.g.f. for row n of R.