<u>Asymptotic approximations for some prime multiplets</u>

Let Π(m,n) be the frequency of prime multiplets p≤n.
m=1: (p) → prime
m=2: (p, p+2) →twin prime
m=3: (p, p+2, p+6), type 1 or (p, p+4, p+6), type 2→prime triplet
m=4: (p, p+2, p+6, p+8)→prime quadruplet
m=5: (p, p+2, p+6, p+8, p+12) , type 1
or (p, p+4, p+6, p+10, p+12) , type 2→prime quintuplet

H. Hardy and J. E. Littlewood first published, in 1923, several conjectures. One of them says that Π(m,n) is asymptotically equal to $\Pi^*(m,n)$ in formula (1). In the appendix I will give an elementary deduction of the formula (1) and the coefficients (2):

$$(1)\ \Pi(m,n) \simeq \Pi^*(m,n) = c(m) \int_2^n \frac{dt}{(\log t)^m} \text{ with}$$

$$(2)\ c(1)=1,\ c(m) = b(m) \prod_{p>m} \left(1-\frac{m-1}{p-1}\right)\left(1+\frac{1}{p-1}\right)^{m-1} \text{ for } m>1$$

and b(2)=2, b(3)=9, b(4)= $^{27}/_{2}$, b(5)= $(¼)^5 \cdot 15^4$

$$(3)\ c(2) = 2 \prod_{k=2}^{\infty} \left(1-\frac{1}{(p_k-1)^2}\right) = 1.32032363 \text{ according to Hardy and Littlewood.}$$

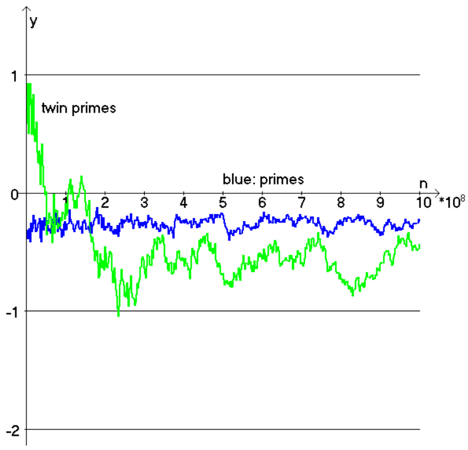
c(3) = 5.71649719, c(4) = 4.15118086, c(5) = 20.2635899

The table shows that $\Pi^*(m,n)$ is a good approximation for Π(m,n) up to n = 10^9:

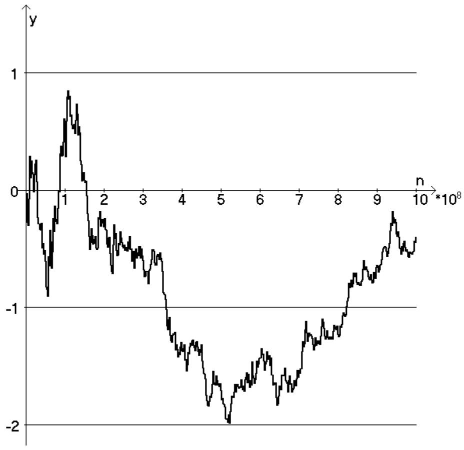
n	Π(2,n)	Π*(2,n)	Π(3,n)	Π*(3,n)	Π(4,n)	Π*(4,n)	Π(5,n)	Π*(5,n)
10^8	440311	440368	111156	110982	4767	4735	1383	1422
$2\cdot10^8$	813370	813779	196836	196975	8096	8057	2264	2285
$3\cdot10^8$	1166479	1167169	275821	276136	10972	11031	3002	3036
$4\cdot10^8$	1507732	1508435	350443	351257	13712	13804	3670	3724
$5\cdot10^8$	1840169	1841093	422440	423553	16330	16440	4309	4370
$6\cdot10^8$	2166300	2167124	492692	493699	18838	18971	4938	4984
$7\cdot10^8$	2486867	2487794	560968	562122	21274	21420	5540	5573
$8\cdot10^8$	2802750	2803980	628138	629114	23659	23800	6126	6142
$9\cdot10^8$	3115261	3116322	694355	694888	26081	26123	6700	6693
10^9	3424505	3425308	759256	759606	28387	28397	7221	7230

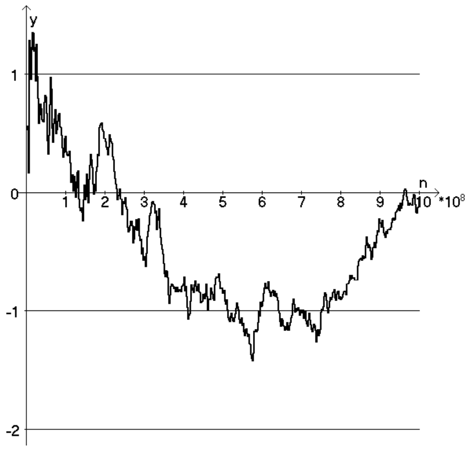
The formulas can be based on a stochastic conjecture, see (4.3):
(4.1) The formula is proven for m=1: The asymptotic density of primes is f(n)= 1/log(n).
(4.2) A random variable r(n) taking the value 1 with this probability (and 0 else) creates the same asymptotic distribution of the cumulated variable (expected value) as another variable q(n) with q(n)= 1 / 0 if n is prime / not prime. By using a sieve and so excluding as many non-primes as possible we can calculate the probability that a pair (n,n+2) or another multiplet is randomly selected.
(4.3) The conjecture is that we can re-interpret this probability density as a “true” density and so find the formulas above for the distribution of prime multiplets.
Moreover, we can compare the difference s = Π(m,n) - $\Pi^*(m,n)$ with the standard deviation $\sigma = \sqrt{\Pi^*(m,n)}$ (because of f(n)<<1).

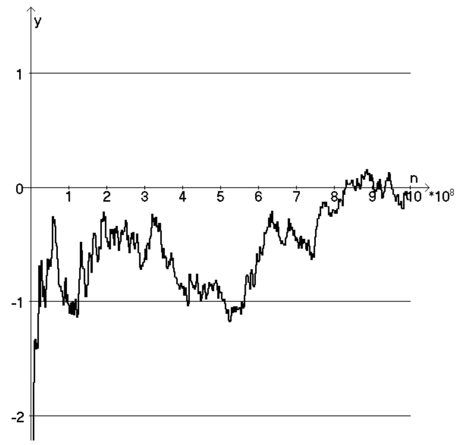
Visualization of the deviation of $\Pi(m,n)$ from $\Pi^*(m,n)$:
(y-axis: unit σ   x-axis: $n \leq 10^9$)



m=1,2: primes and twins


m=3: prime triplets


m=4: prime quadruplets


m=5: prime quintuplets

<u>Appendix: Deduction of the formula (1) and the coefficients (2):</u>

<u>k-primes</u>

Let $p_1=2$, $p_2=3$, .. the sequence of primes. A number x which is prime to any $p_j \leq p_k$ will be called a k-prime. x will be also called a (m,k)-multiplet for m>1 :

m=2: k-twin, if x and x+2 are k-primes

m=3: k-triplet, if x, x+2 (4) and x+6 are k-primes, type 1 (2)

m=4: k-quadruplet, if x, x+2, x+6 and x+8 are k-primes

m=5: k-quintuplet, if x, x+2 (4), x+6, x+8 (10) and x+12 are k-primes, type 1 (2)

Note: k-triplets and k-quintuplets of type 2 are included later, see (A6).

Let Q(m,k), k>1, be the sequence of (m,k)-multiplets.

Examples for k≤3:

(A1) Q(1,1)= (3,5,7,9,11,..), odd numbers.

Removing the multiples of $p_2=3$ we obtain

(A2)Q(1,2)= (5,7,11,13,17,19,23,25,29,31,35, ..)

This sequence can be split up into two arithmetic progressions (5,11,17,23, ..) and (7,13,19,25, ..) with the difference $d_2=2\cdot3=6$.

Each of them can be split up into 5 subsequences (difference $d_3 = 2\cdot3\cdot5 = 30$): By removing the multiples of $p_3=5$, eight progressions remain in Q(1,3), three (11,.. 17,.. 29,..) in Q(2,3), two (11,.. 17,..) in Q(3,3) and one (11,..) in Q(4,3) as well as in Q(5,3). See right table.

5	35	65	…
7	37	67	…
11	41	71	…
13	43	73	…
17	47	77	…
19	49	79	…
23	53	83	…
25	55	85	…
29	59	89	…
31	61	91	…

Q(m,k) is the union of q(m,k) arithmetic progressions with the difference $d_k = \prod_{j=1}^{k} p_j$. Here are some values of q(m,k):

(A3) q(1,1)=1, q(1,2)=2, q(1,3)=8, q(2,2)=q(3,2)=q(4,2)=q(5,3)=1

<u>Recurrence for q(m,k):</u>

m=1
Any arithmetic progression in Q(1,k-1) with the difference d_{k-1} can be split up into p_k progressions with the difference $d_k = p_k\cdot d_{k-1}$. They belong to different residue classes (mod p_k) because p_k and d_{k-1} are relatively prime. By removing the progression representing the class 0 (mod p_k) in Q(1,k) we erase one k-prime and obtain
$q(1,k) = q(1,k-1)\cdot (p_k-1)$.

m=2
Each (k-1)-twin belongs to a pair of progressions in Q(2,k-1). Executing the step k-1→k we remove the 2 subsequences with 0 (mod p_k). This way two k-twins are erased with the result: $q(2,k) = q(2,k-1)\cdot (p_k-2)$.

m≤5 (A4) Generalization: $q(m,k) = q(m,k-1)\cdot (p_k-m)$.
There seems to be a problem with m=3. K-triples do overlap when they are part of a quintuple. Then only 5 (instead of 2·3=6) subsequences with 0 (mod p_k) are removed. But with the subsequence belonging to the overlapping number two k-triples are erased so that recurrence (A4) is correct for m=3. It also holds for m=5 in the case of overlapping quintuples.

The recurrence (A4) is valid for $p_k>m$, i.e. $k>k_m$ with $k_1 = 1$, $k_2 = k_3 = k_4 = 2$, $k_5 = 3$

<u>Recurrence for the density δ(m,k) of Q(m,k):</u>

Generally: The density of a sequence, being the union of n (n≤d) arithmetic progressions with the difference d, is $^n/_d$.

$$\delta(m,k) = \frac{q(m,k)}{d_k} = \frac{q(m,k-1)(p_k-m)}{p_k\cdot d_{k-1}} = \delta(m,k-1)\frac{p_k-m}{p_k}$$

Explicit version: (A5) $\delta(m,k) = \delta(m, k_m)\prod_{m<p\le p_k}\frac{p-m}{p}$, p prime

<u>Inclusion of k-triples and k-quintuples of type 2.</u>

Their number is the same as of type 1. So we replace q(m,k) by 2q(m,k) for m=3, 5.
This leads to the basic densities $\delta(m, k_m)$:
(A6) δ(1,1) = ½, δ(2,2)= $^1/_6$, δ(3,2)= $^2/_6$, δ(4,2)= $^1/_6$, δ(5,3)= $^2/_{30}$
and δ(1,2)= $^2/_6$, δ(1,3)= $^8/_{30}$

A stochastic approach to a conjecture

Let u(m,n) be the asymptotic density of prime multiplets. For primes there are well known formulas:

$$\text{(A7) } u(1,n)=\frac{1}{\log n} \quad \text{and} \quad \Pi(1,n) \simeq \int_2^n \frac{dt}{\log t} = \mathrm{Li}(n).$$

$w(m,k,n) := \frac{u(m,n)}{\delta(m,k)}$ can be thought of as the probability that a randomly selected (m,k)-multiplet is a prime multiplet.

Conjecture: The events “n is a prime” and “n+x is a prime” are unrelated for any n, n+x Є Q(1,k). Then the probability that both events occur is, w(1,k,n)· w(1,k,n+x) or, for x=2 and large n: $w(2,k,n) = w(1,k,n)^2$ or generally for multiplets:

$$\text{(A8) } w(m,k,n) = w(1,k,n)^m$$

$$\frac{u(m,n)}{\delta(m,k)} = \left(\frac{u(1,n)}{\delta(1,k)}\right)^m \rightarrow u(m,n) = \frac{\alpha(m,k)}{\log^m n} \text{ with } \alpha(m,k) = \frac{\delta(m,k)}{\delta(1,k)^m}$$

The concentration of primes in Q(m,k) increases with k, and so the transition k→∞ is reasonable. With $c(m) := \lim_{k\to\infty} \alpha(m,k)$ the conjecture is

$$\text{(A9) } u(m,n)= \frac{c(m)}{\log^m n}$$

Coefficients used in (1) and (2)

$$\alpha(m,k) = \frac{\delta(m,k)}{\delta(1,k)^m} = b(m) \prod_{m<p\le p_k} \frac{p-m}{p}\left(\frac{p}{p-1}\right)^m = b(m) \prod_{m<p\le p_k} \left(1-\frac{m-1}{p-1}\right)\left(1+\frac{1}{p-1}\right)^{m-1}, \text{ p prime}$$

with $b(m) = \frac{\delta(m,k_m)}{\delta(1,k_m)^m}$ (for these special values see (A6)).

Result: $b(2)= \frac{\delta(2,1)}{\delta(1,1)^2} = 2$, $b(3)= \frac{\delta(3,2)}{\delta(1,2)^3} = 9$, $b(4)= \frac{\delta(4,2)}{\delta(1,2)^4} = \frac{27}{2}$, $b(5)= \frac{\delta(5,3)}{\delta(1,3)^5} = \frac{1}{4}\left(\frac{15}{4}\right)^4$