A NOTE ON STEPHAN'S CONJECTURE 87

ELIZABETH WILMER

Recently Stephan [5] posted 117 conjectures based on an extensive analysis of the On-line Encyclopedia of Integer Sequences [3, 4]. Here we prove conjecture 87.

Let $K_{n,m}$ denote a complete bipartite graph with part sizes n and m, and let P_n denote a path with n vertices. Fix an integer $k > 1$. Here we are concerned with counting perfect matchings on the graphs $G_n = K_{1,k-1} \times P_{2n}$. (For $k > 2$, there are no perfect matchings on $K_{1,k-1} \times P_{2n+1}$, which is bipartite with parts of unequal size.)

For $k = 2$ and $k = 3$, this problem is equivalent to the extremely well-studied problems of counting domino tilings of a 2-by-$2n$ or 3-by-$2n$ grid, respectively. See [2], section 7.1, for an extensive discussion.

We give two versions of the central combinatorial argument (Lemma 1 and the first proof of Lemma 3). The second argument, whose form was suggested by Henry Cohn, is simpler. However, the work of the first yields some information on the structure of the matchings (Corollary 2). The two recurrences derived are easily seen to be equivalent (second proof of Lemma 3), so we follow only one path to the generating function (Proposition 4).

Let's name the vertices of G_n. Call $c_1, c_2, \ldots, c_{2n}$ the *centers*, and call $d_{i,j}$, $1 \le i \le 2n$, $1 \le j \le k-1$ be the *peripheral vertices*. There two types of edges:

- Horizontal: $\{c_i, d_{i,j}\}$, for $i = 1, \ldots, 2n$ and $j = 1, \ldots, k-1$.
- Vertical: $\{c_i, c_{i+1}\}$ and $\{d_{i,j}, d_{i+1,j}\}$ for $1 \le i \le 2n-1$, $1 \le j \le k-1$. We say $\{c_i, c_{i+1}\}$ and $\{d_{i,j}, d_{i+1,j}\}$ are *at level i*.)

Lemma 1. *Let $k > 1$ be a positive integer and let a_n be the number of perfect matchings of the graph G_n. Then*

$$a_0 = 1, \qquad a_1 = k,$$

and

$$a_n = ka_{n-1} + (k-1)(a_{n-2} + a_{n-3} + \cdots + a_0)$$

for $n \ge 2$.

Proof. Because G_0 has a single (null) matching, $a_0 = 1$.

How many matchings does G_1 have? Consider c_2. If it is matched with c_1, then every peripheral vertex must be matched via a vertical edge. If, on the other hand, c_2 is matched to some $d_{2,j}$ via a horizontal edge, then c_1 must be matched with $d_{1,j}$ (for the same j!), and all other peripheral vertices are matched via vertical edges. In every case, once we match c_2, the rest of the matching is determined. Since c_2 has degree k, we have $a_1 = k$.

Now consider G_n for some $n \ge 2$. If a matching contains no vertical edges at level $2n-2$, it consists of a matching of G_{n-1}, together with a matching of G_1—the

Date: October 15, 2004.

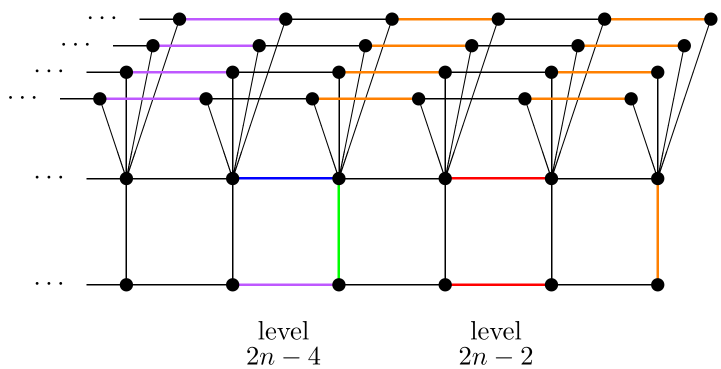

FIGURE 1. Sketch of a matching with a link at level $2n-2$. The red (link) edges force all the orange edges. The matching must include either the green edge or the blue edge. If green, then there are no vertical edges at level $2n-4$. If blue, then all the purple edges are forced, so there is a link *in the same page* at level $2n-4$.

latter covers the vertices at levels $2n-1$ and $2n$. Clearly, there are $a_{n-1}a_1 = ka_{n-1}$ such matchings.

What if there are vertical edges at level $2n-2$? There cannot be only one; if so, then the remaining $2k-1$ vertices at levels $2n-1$ and $2n$ must be matched with each other. However, the number of such vertices is odd.

There also cannot be two vertical edges at level $2n-2$ that are both peripheral. If there were, say $\{d_{2n-2,r}, d_{2n-1,r}\}$ and $\{d_{2n-2,s}, d_{2n-1,s}\}$, then both $\{d_{2n,r}, c_{2n}\}$ and $\{d_{2n,s}, c_{2n}\}$ would have to be in the matching, which is impossible.

The only remaining possibility is that there are exactly two vertical edges at level $2n-2$ in the matching, one of which is central and one of which is peripheral: $\{c_{2n-2}, c_{2n-1}\}$ and $\{d_{2n-2,r}, d_{2n-1,r}\}$ for some $1 \leq r \leq k-1$. When this happens, we say that there is a *link at level* $2n-2$ *in page* r.

When there is a link at level $2n-2$ in page r, the configuration of the matching at higher levels is completely determined: the horizonal edge $\{c_{2n}, d_{2n,r}\}$ must be present, as must be all peripheral vertical edges at level $2n-1$ not in page r. Similarly, all peripheral vertical edges at level $2n-3$, but not in page r, must be present. (See Figure 1, where the link edges are shown in red and the forced edges are shown in orange.)

Now consider c_{2n-3}. It must be matched with either $d_{2n-3,r}$ (shown in green in Figure 1) or c_{2n-4} (shown in blue in Figure 1). In the former case, no vertical edges at level $2n-4$ can be used in the matching. Hence the entire matching splits into two pieces: a matching of G_{n-2}, along with a configuration of the type shown in Figure 1 in levels $2n-3$ and up. Since there are $k-1$ possible pages in which to place the link at level $2n-2$, there are $(k-1)a_{n-2}$ such matchings.

If, instead, we join c_{2n-3} to c_{2n-4}, then $d_{2n-3,r}$ must be matched to $d_{2n-4,r}$, and we must have a link at level $2n-4$ in page r. As before, no other vertical edges at level $2n-4$ can be used, but all vertical edges outside of page r are forced at level $2n-5$ (see the purple edges in Figure 1).

Now, vertex c_{2n-5} must be matched either horizontally or vertically. There will be $(k-1)a_{n-3}$ matchings in the former category. For the latter, there will be a link forced at level $2n-6$, and so on...

To sum up: every matching of G_n will have a solidly linked section starting from level $2n$ and running down to the first even level $2m$ where there are no vertical edges present.

- When $m = n-1$, then, as we have noted, there are ka_{n-1} possible matchings.
- When $1 \leq m \leq n-2$, there are a_m ways to match the initial segment, and $k-1$ ways to build the solidly linked segment, for a total of $(k-1)a_m$ matchings.
- When every even level is linked, there are $k-1 = (k-1)a_0$ possible matchings.

Hence

$$a_n = ka_{n-1} + (k-1)(a_{n-2} + \cdots + a_0).$$

□

Corollary 2. *Every perfect matching on G_n has the following properties:*

1. *No more than one edge from any horizontal level can be included.*
2. *At least $k-2$ vertical edges from each odd level must be included.*
3. *From each even level, either zero or two vertical edges can be included. If two vertical edges from an even level are present in the matching, one is central and the other is peripheral.*
4. *If vertical edges from consecutive even levels are included, they must lie in the same page of the graph.*

Proof. By strong induction on n. The final linked segment of the matching (which was the key to to the proof of Lemma 1) has all these properties; so (by the inductive hypothesis) must the rest of the matching. □

The second combinatorial argument, while closely related (of course!) to the first argument, is simpler; it also directly derives the recurrence relation we want. The many-to-one structure (which is illustrated beautifully clearly in the proof of identity 7 of [1]) was suggested by Henry Cohn.

Lemma 3. *Let $k > 1$ be a positive integer and let a_n be the number of perfect matchings of the graph G_n. Then*

$$a_0 = 1, \qquad a_1 = k,$$

and

$$a_n = (k+1)a_{n-1} - a_{n-2}.$$

for $n \geq 2$.

Proof 1 (combinatorial). The computations for $a_0 = 1$ and $a_1 = k$ go through as before, of course. Now assume that $n \geq 2$. We will build a correspondence between

- a multiset containing $k+1$ copies of each matching on G_{n-1}, and
- the set containing all matchings on both G_n and G_{n-2}.

Once we have done so, it will immediately follow that

$$(k+1)a_{n-1} = a_n + a_{n-2}.$$

Now, on to the correspondence. Since each matching on G_{n-1} can be extended to a matching of G_n by appending any of the k matchings on G_1, we can match k of the

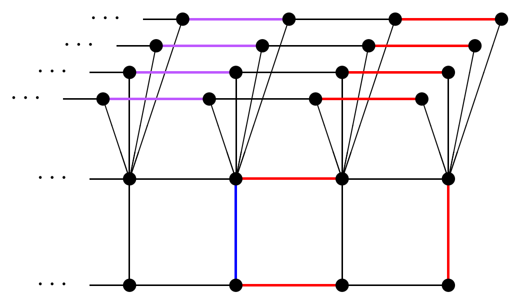

FIGURE 2. To construct a matching of G_n, linked at level $2n-2$, from a matching of G_{n-1} with a horizontal edge (blue) at level $2n-2$, delete the horizontal edge (whose presence forces many verticals at level $2n-3$, shown in purple) and add in the red edges.

copies of each matching of G_{n-1} to a matching of G_n *with no link at level* $2n-2$. Furthermore, every such matching on G_n arises (exactly once) in this fashion.

What about the remaining (single) copies of the matchings of G_{n-1}? These must be made to correspond to the matchings on G_{n-2} and the matchings on G_n which are linked at level $2n-2$.

Given a matching on G_{n-1}: if it contains all vertical edges at level $2n-3$, deleting those edges yields a matching of G_{n-2}. Furthermore, every matching of G_{n-2} arises (exactly once) in this fashion.

Otherwise, the matching on G_{n-1} must contain (exactly one) horizontal edge at level $2n-2$, say, $\{c_{2n-2}, d_{2n-2,r}\}$. Build a new matching on G_n as follows: include all edges of the original matching *except* $\{c_{2n-2}, d_{2n-2,r}\}$. Add in edges $\{c_{2n-2}, c_{2n-1}\}$, $\{d_{2n-2,r}, d_{2n-1,r}\}$, $\{c_{2n}, d_{2n,r}\}$, and, for every $s \neq r$, $\{d_{2n-1,s}, d_{2n,s}\}$. (See Figure 2.)

The resulting matching on G_n is linked at level $2n-2$. Furthermore, every matching of G_{n+1} linked at level $2n$ arises (exactly once) in this construction. □

Proof 2 (from Lemma 1). Fix $n \geq 2$. Then

$$\begin{aligned} a_n &= ka_{n-1} + (k-1)(a_{n-2} + \cdots + a_0) \\ &= ka_{n-1} + ka_{n-2} + (k-1)(a_{n-3} + \cdots + a_0) - a_{n-2} \\ &= ka_{n-1} + a_{n-1} - a_{n-2}. \end{aligned}$$

□

Proposition 4 (Conjecture 87)**.** *Let $k > 1$ be a positive integer, let a_n be the number of perfect matchings of the graph $K_{1,k-1} \times P_{2n}$, and let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then*

$$A(x) = \frac{1-x}{1-(k+1)x+x^2}.$$

Proof. As Wilf enthusiastically recommends ([6], Chapter 1), we multiply each equation listed in Lemma 3 by x^n, then sum over n, obtaining

$$\begin{aligned} A(x) &= 1 + kx + \sum_{n=2}^{\infty}(k+1)a_{n-1}x^n - \sum_{n=2}^{\infty} a_{n-2}x^n \\ &= 1 + kx + (k+1)x(A(x)-1) - x^2A(x) \\ &= 1 - x + A(x)\left((k+1)x - x^2\right). \end{aligned}$$

Solving for $A(x)$ yields the claimed formula. □

References

[1] Benjamin, Arthur T. and Quinn, Jennifer J. *Proofs that Really Count: The Art of Combinatorial Proof.* Mathematical Association of America, Washington, D.C., 2003.

[2] Graham, Ronald L., Knuth, Donald E., and Patashnik, Oren. *Concrete Mathematics.* Addison-Wesley, Reading, 1994.

[3] Sloane, N. J. A. The On-Line Encyclopedia of Integer Sequences, published electronically at `http://www.research.att.com/ñjas/sequences/`, 2004.

[4] Sloane, N. J. A. The on-line encyclopedia of integer sequences. *Notices Amer. Math. Soc.* **50** (2003), pp. 912–915.

[5] Stephan, Ralf. Prove or Disprove: 100 Conjectures from the OEIS. arXiv:math.CO/0409509

[6] Wilf, Herbert. *generatingfunctionology.* Academic Press, San Diego, 1990.

Department of Mathematics, Oberlin College, Oberlin, OH 44074
E-mail address: `elizabeth.wilmer@oberlin.edu`