On a generalization of Euler-Gauss formula for the Gamma function

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to Matthieu and Thomas

Abstract

The well-known Euler-Gauss formula for the *Gamma* function:

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!n^z}{(z+1)(z+2)...(z+n)} \quad \textbf{[0]}$$

where:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt$$

may be regarded as a limit involving a recursive sequence of order 1. Indeed, defining ***t*** $=(t_n)_{n\geq 1}$ as follows:

$$t_1 = 1 \quad t_{n+1} = \frac{n+z}{n} t_n$$

then we get:

$$\lim_{n\to\infty} \frac{n^z}{t_n} = \Gamma(z+1).$$

In this note we consider recursive sequences of higher order yielding the *Gamma* function in a similar way. First, we will present 2 symmetrical recursive sequences of order 2 (which are an extension of an earlier result) and corollaries allowing us to have aesthetic relations between *e* and *π*. Then we will discuss 2 generalizations for both sequences at any order, which are themselves derived from a more general property of the *Gamma* function (as we will see in the last section). This study led to an infinite set of constants where *e*, *π* and γ are again strongly connected. Proofs are omitted in this long presentation and include matrix relations, theorems on asymptotic behaviour of coefficients of certain generating functions, as well as known relations for certain values of *Gamma* and *Digamma* functions evaluated at rational arguments. It is relevant to this study to determine precisely the behaviour of linear recursions with varying coefficients at arbitrary order.

A. *The order 2*

1) Two symmetrical recursive sequences

Let $z = z_1 + z_2$ be any suitable complex number (i.e. not a negative integer) and $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ be the 2 recursions defined "symmetrically" as follows:

$u_1 = v_1 = 0$, $u_2 = v_2 = 1$

$$u_{n+2} = \frac{n+z_1}{n}\ u_{n+1} + \frac{z_2}{n}\ u_n$$

$$v_{n+2} = \frac{2z_1}{n}\ v_{n+1} + \frac{n+2z_2}{n}\ v_n$$

then for any suitable z_1 and z_2 we get the "dual" formulas:

$$\lim_{n\to\infty} \frac{n^z}{u_n} = e^{z_2} \times \Gamma(z+1)$$

$$\lim_{n\to\infty} \frac{n^z}{2v_n} = 2^{z_2 - z_1} \times \Gamma(z+1)$$

2) *e* and π in a mirror

The above sequences of order 2 were already described for specific values in **[1] [2] [3] [4] [5]**. That was $(z_1, z_2) = (0,1)$ for ***u*** and $(z_1, z_2) = (1/2, 0)$ for ***v***. Since $\Gamma(1/2) = \sqrt{\pi}$, under those conditions, we have the simple mirror relationship between the 2 famous constants:

$u_1 = v_1 = 0$, $u_2 = v_2 = 1$

$$u_{n+2} = u_{n+1} + \frac{u_n}{n}$$

$$v_{n+2} = \frac{v_{n+1}}{n} + v_n$$

and then we get:

$$\lim_{n\to\infty} \frac{n}{u_n} = e$$

$$\lim_{n\to\infty} \frac{2n}{v_n^{\ 2}} = \pi$$

3) A pretty "discrete" generalization

Let $m \geq 0$ be an integer and $\boldsymbol{u}$ and $\boldsymbol{v}$ be defined as follows:

$u_1 = v_1 = 0$, $u_2 = v_2 = 1$

$$u_{n+2} = u_{n+1} + \frac{u_n}{n+m}$$

$$v_{n+2} = \frac{v_{n+1}}{n+m} + v_n$$

let now $\boldsymbol{f}$ and $\boldsymbol{g}$ denote the following functions of *m:* $f(m) = \lim_{n\to\infty} \frac{u_n}{n}$ and $g(m) = \lim_{n\to\infty} \frac{v_n}{\sqrt{n}}$ so that $f(0) = 1/e$ and $g(0) = \sqrt{2/\pi}$ as we see above. Then there are 2 integer sequences $\boldsymbol{a}$ = $(a_k)_{k\geq 0}$, $\boldsymbol{b}$ = $(b_k)_{k\geq 0}$ and 2 rational sequences $\boldsymbol{p}$ = $(p_k)_{k\geq 0}$, $\boldsymbol{q}$ = $(q_k)_{k\geq 0}$ such that we get the twin formulas:

$$(-1)^m \frac{f(m)}{f(0)} = a_m - e \times b_m$$

$$(-1)^m \frac{g(m)}{g(0)} = p_m - \pi \times q_m$$

which seems to be an elegant relation between e and π. More precisely, we have:

$$a_k = (k+1)! \qquad b_k = (k+1)!\sum_{i=0}^{k+1} \frac{(-1)^i}{i!} = \left\lfloor \frac{(k+1)!}{e} + \frac{1}{2} \right\rfloor$$

and

$$q_{2k} = q_{2k+1} = k\frac{C_{2k}^k}{4^k} \qquad p_0 = 1 \text{ and } k \geq 1 \;\; p_k = \frac{k}{2q_k}$$

It is possible to obtain formulas for various starting values. For example, if $m = 0$ and:

$u_1 = v_1 = x$, $u_2 = v_2 = y$

$$u_{n+2} = u_{n+1} + \frac{u_n}{n}$$

$$v_{n+2} = \frac{v_{n+1}}{n} + v_n$$

Then we get:

$$\lim_{n\to\infty}\frac{u_n}{n}=e^{-1}\times\left(x\times(e-2)+y\right)\qquad \lim_{n\to\infty}\frac{v_n}{\sqrt{n}}=\sqrt{\frac{2}{\pi}}\times\left(x\times\left(\frac{\pi}{2}-1\right)+y\right)$$

In general, for any integer *m*, if $\left\{\alpha\right\}$ denotes the fractional part of α, there is a closed form formula for ***u:***

$$\lim_{n\to\infty}\frac{u_n}{n}=x\times\left(\frac{1+(-1)^m}{2}-(-1)^m\left\{\frac{(m+2)m!}{e}\right\}\right)+y\times\left(\frac{1-(-1)^m}{2}+(-1)^m\left\{\frac{(m+1)!}{e}\right\}\right)$$

and regarding ***v***:

$$\lim_{n\to\infty}\frac{v_n}{\sqrt{n}}=(-1)^m\times\sqrt{\frac{2}{\pi}}\left(x\times(p_{m+1}-\pi\times q_{m+1})+y\times(p_m-\pi\times q_m)\right)$$

4) <u>Gosper's continued fraction</u>

Aware of an earlier draft mentioning this ***v*** recursion, Bill Gosper **[6]** promptly derived an infinite product of matrices leading to the following weird continued fraction:

$$\frac{2\sqrt{2}}{\pi-2}=\frac{\sqrt{2/1}}{1}+\cfrac{\sqrt{3/1}}{\frac{\sqrt{3/2}}{2}+\cfrac{\sqrt{4/2}}{\frac{\sqrt{4/3}}{3}+\cfrac{\sqrt{5/3}}{\frac{\sqrt{5/4}}{4}+\dots}}}$$

5) <u>The « low » generalization</u>

We would also like to mention a function *E* derived from the following ***u***-type recursion $u_1=0$, $u_2=1$, $u_{n+2}=u_{n+1}+\dfrac{u_n}{n+z}$ and then: $E(z)=\lim_{n\to\infty}u_{n+1}-u_n$. This function *E* satisfies:

$$E(z-1)=1-\frac{z}{e}\times\gamma(z,1)$$

where $\gamma(z,1)$ is the lower incomplete *Gamma* function: $\gamma(z,1)=\int_0^1 e^u u^{z-1}du$ **[7].**

E also satisfies the functional equation $E(z+1)+(z+2)\,E(z)=1$; but I won't digress here on this kind of "low" generalization, preferring the "high" one as we will see in the next sections.

B. *u and v at any order*

Remarkably, sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ can be generalized further. Specifically, there are 2 "symmetrical" recursive sequences at any order, yielding the *Gamma* function in a similar way as for order 2.

To do this, let $r \geq 2$ be an integer and let $z = z_1 + z_2 + \ldots + z_r$ be any suitable complex number (i.e. not a negative integer), and define 2 "symmetrical" recursions $\boldsymbol{u}$ and $\boldsymbol{v}$, both of order r, as follows:

$u_r = v_r = 1 \qquad u_k = v_k = 0$ for $1 \leq k < r$

$$u_{n+r} = u_{n+r-1} + \frac{1}{n}\left(z_1 u_{n+r-1} + \ldots + z_r u_n\right)$$

$$v_{n+r} = \frac{r}{n}\left(z_1 v_{n+r-1} + \ldots\ldots\ldots + z_r v_n\right) + v_n$$

Then there are 2 families of positive real constants $\left(\lambda_{r,i}\right)_{1 \leq i \leq r}$ and $\left(\mu_{r,i}\right)_{1 \leq i \leq r}$ such that for any suitable $(z_i)_{1 \leq i \leq r}$ we get the striking formulas:

$$\lim_{n\to\infty} \frac{n^z}{u_n} = \left(\lambda_{r,1}{}^{z_1} \times \lambda_{r,2}{}^{z_2} \times \ldots \times \lambda_{r,r}{}^{z_r} \right) \times \Gamma(z+1)$$

$$\lim_{n\to\infty} \frac{n^z}{r v_n} = \left(\mu_{r,1}{}^{z_1} \times \mu_{r,2}{}^{z_2} \times \ldots \times \mu_{r,r}{}^{z_r} \right) \times \Gamma(z+1)$$

There is a big difference here: $\left(\lambda_{r,i}\right)_{1 \leq i \leq r}$ don't depend on r while $\left(\mu_{r,i}\right)_{1 \leq i \leq r}$ depend on r.

Namely:

$$\lambda_{r,i} = e^{H_{i-1}}$$

where $H_k = \sum_{i=1}^{k} 1/i$ is the k-th harmonic number with $H_0 = 0$.

Unfortunately, despite this apparent symmetry, it was not so immediate to derive a general formula for $\left(\mu_{r,i}\right)_{1 \leq i \leq r}$. By the way, those important properties for $\left(\mu_{r,i}\right)_{1 \leq i \leq r}$ suggested more was true:

$$\mu_{r,1} < \mu_{r,2} < \ldots < \mu_{r,r} = r$$

$$\mu_1 \times \mu_2 \times \ldots \times \mu_r = 1$$

$$\mu_{\alpha r, \alpha \mathrm{i}} = \alpha\, \mu_{r, \mathrm{i}}$$

Indeed, those properties come from a much more general result described subsequently. Note that the case $r = 2$ is easily determined since then $\mu_{2,1} = 1/2$ and $\mu_{2,2} = 2$.

Remark:

It can be shown that $(v_n)_{n>0}$ generates integer sequences for some values of $(z_1, z_2, .., z_r)$, as in the case $(z_1, z_2, .., z_r) = (1, 1, ..., 1, m)$, m integer. An interesting example of order 2 is $(z_1, z_2) = (1, 2)$ yielding sequence n°A006918 in **[8]**. Also, $(z_1, z_2) = (2, 2)$ gives sequence n°A001752, and $(z_1, z_2, z_3) = (1, 1, 1)$ gives sequence n°A014125. This is a consequence of the generating functions; e.g.,

for $(z_1, z_2, z_3) = (1, 1, m)$ $\boldsymbol{v}$ sequence has *g.f.*:

$$\frac{1}{(1-x)^3 (1-x^3)^m}$$

and for $(z_1, z_2, z_3) = (m/3, m/3, m/3)$ *g.f.* is:

$$\frac{1}{(1-x)^m (1-x^3)}.$$

C. *The order 3: an unexpected appearance of π*

Order 3 for ***u*** and ***v*** sequences gives:

$u_3 = v_3 = 1 \qquad u_k = v_k = 0 \text{ for } 1 \le k \le 2$

$$u_{n+3} = u_{n+2} + \frac{z_1 u_{n+2} + z_2 u_{n+1} + z_3 u_n}{n}$$

$$v_{n+3} = 3 \times \frac{z_1 v_{n+2} + z_2 v_{n+1} + z_3 v_n}{n} + v_n$$

and then, from the previous section, for any suitable (z_1, z_2, z_3):

$$\lim_{n\to\infty} \frac{n^z}{u_n} = e^{\frac{2z_2+3z_3}{2}} \times \Gamma(z+1)$$

$$\lim_{n\to\infty} \frac{n^z}{3v_n} = c^{z_2-z_1} \times 3^{z_3-z_1} \times \Gamma(z+1)$$

thus an interesting constant arises: c = 1.4298843084…. with the amazing closed form originally conjectured by Paul D. Hanna:

$$c = \frac{1}{\sqrt{3}} e^{\frac{\pi}{\sqrt{12}}}$$

For order r >3, let's go to the next section where we will see the reason why it is not so unexpected.

D. *The generalized Euler-Gauss formula*

The *Gamma* function obeys a much more general rule. We will mention the main result and the last possible recursion of order 3, providing a complete set of associated constants in that case. Finally, we will introduce an infinite table of constants possessing astonishing properties. We will provide a closed form formula for them, revealing a part of the mystery of the *Gamma* function. Computation played a great role here, and we believe that mathematics by experiments is indeed plausible reasoning in the 21-*st* century **[9].**

1) The generalized Euler-Gauss formula

Let $r \geq 1$ and $1 \leq s \leq r$ be fixed integers, let $(z, z_i)_{1\leq i \leq r}$ be any suitable complex numbers ($\mathrm{re}(z) > 0$, $\mathrm{re}(z_i) > 0$ for all $1 \leq i \leq r$ are sufficient conditions) satisfying $z = z_1 + z_2 + \ldots + z_r$.

Define the recursion ***w*** of order *r*, depending on *s*, as follows:

$w_r = 1 \qquad w_k = 0 \text{ for } 1 \leq k < r$

$$w_{n+r} = w_{n+r-s} + \frac{s}{n} \times \left(z_1 w_{n+r-1} + z_2 w_{n+r-2} + \ldots + z_r w_n \right)$$

Then there are *r* positive real constants $\left(\sigma_{r,s,i} \right)_{1\leq i\leq r}$ such that we get:

$$\lim_{n\to\infty} \frac{n^z}{s\, w_n} = \left(\sigma_{r,s,1}{}^{z_1} \times \sigma_{r,s,2}{}^{z_2} \times \ldots \times \sigma_{r,s,r}{}^{z_r} \right) \times \Gamma(z+1)$$

The previous section dealt with the special cases $s = 1$ and $s = r$, giving ***u*** and ***v*,** respectively.

2) The last case of order 3

Taking $s = 2$ gives:

$w_1 = w_2 = 0 \quad w_3 = 1$

$$w_{n+3} = w_{n+1} + 2 \times \frac{z_1 w_{n+2} + z_2 w_{n+1} + z_3 w_n}{n}$$

which yields for suitable z_i :

$$\lim_{n\to\infty} \frac{n^z}{2 w_n} = \left(\sigma_{3,1,1}{}^{z_1} \times \sigma_{3,1,2}{}^{z_2} \times \sigma_{3,1,3}{}^{z_3} \right) \times \Gamma(z+1)$$

where:

$$\sigma_{3,1,1} = \frac{1}{2} \qquad \sigma_{3,1,2} = 2 \qquad \sigma_{3,1,3} = \frac{1}{2} e^2$$

3) The table of constants

We saw that the 3 possible recursions of order 3 for s = 1, 2, and 3, involve 9 constants. Therefore it is interesting to associate a table of constants as described with the following 3 × 3 array for the constants $\left(\sigma_{3,\mathrm{s},i}\right)_{1\le i\le 3,\,1\le \mathrm{s}\le 3}$:

s \ i	1	2	3
1	1	e	$e^{3/2}$
2	$\frac{1}{2}$	2	$\frac{1}{2}e^2$
3	$\frac{1}{\sqrt{3}}e^{-\pi/\sqrt{12}}$	$\frac{1}{\sqrt{3}}e^{\pi/\sqrt{12}}$	3

Similarly, there is a table of constants for any order r and $1 \le s \le r$. In fact, the table of constants of order r contains the constants of order r-1; thus all constants can be represented with a single infinite array $\left(c_{n,k}\right)_{n\ge 1,\ k\ge 1}$ with $c_{s,i} = \sigma_{r,\mathrm{s},i}$.

The numbers $c(n,k)$ with $(n,k)=1$ and $n \ge k$ are called "primitive" constants since all the others can be obtained from them using the 2 following remarkable rules for any n, k, m positive integers:

$$\mathrm{c}(mn,\, mk) = m\ \mathrm{c}(n,\, k)$$

$$\prod_{i=0}^{m-1} c(m\times n, k+i\times n) = c(n,k)^m$$

2 simple cases of this product identity are:

$$\prod_{i=1}^{n} c(n,i) = 1$$

$$\prod_{i=0}^{m-1} c(mn, ni+1) = e^{mH_{n-1}}$$

Those relations are a consequence of the general formula first suggested by Paul D. Hanna:

$$c(n,k) = ne^{\psi\left(\frac{k}{n}\right)+\gamma}$$

where γ is the Euler's constant **[10]** and $\Psi(z)=\dfrac{\Gamma'(z)}{\Gamma(z)}$ is the *Digamma* function **[11]**.

It is worth noting that for any n $\lim_{k\to\infty}\dfrac{c(n,k)}{k}=e^{\gamma}$.

Thus, one can rewrite the generalized Euler-Gauss formula in **D.1.** as follows:

$$\lim_{n\to\infty}\frac{n^{z}}{w_n}=s^{z+1}\times\Gamma(z+1)\times\exp(\gamma.z)\times\exp\left(\sum_{k=1}^{r}z_k.\Psi(k/s)\right)$$

The table of constants contains nice identities mainly due to the Gauss's *Digamma* theorem **[12],** which allows us to evaluate the *Digamma* function at rational values:

$$\Psi(k/n)+\gamma=-\frac{\pi}{2}\cot\left(\frac{\pi k}{n}\right)-\log(2n)+\sum_{j=1}^{n-1}\left(\cos\frac{2\pi jk}{n}\right)\times\log\left(\sin\frac{j\pi}{n}\right)$$

For instance:

$$c(3,1)=3\times e^{-\frac{\pi}{2}\cot\left(\frac{\pi}{3}\right)-\log(6)+\cos\frac{2\pi}{3}\log\left(\sin\frac{\pi}{3}\right)+\cos\frac{4\pi}{3}\log\left(\sin\frac{2\pi}{3}\right)}=\frac{1}{\sqrt{3}}e^{-\frac{\pi}{\sqrt{12}}}$$

The reflection and duplication formulas are also useful:

$$\Psi(1-x)=\Psi(x)+\pi\cot(\pi x)$$

$$\Psi(2x)=\frac{\Psi(x)+\Psi(x+1/2)}{2}+\log 2$$

The table of constants $(c_{n,k})$ for $1\le n,k\le 8$ is given thereafter (with exact formulas when simple enough) as well as the corresponding numerical table and a graph showing the behaviour of c(n, k) as k grows.

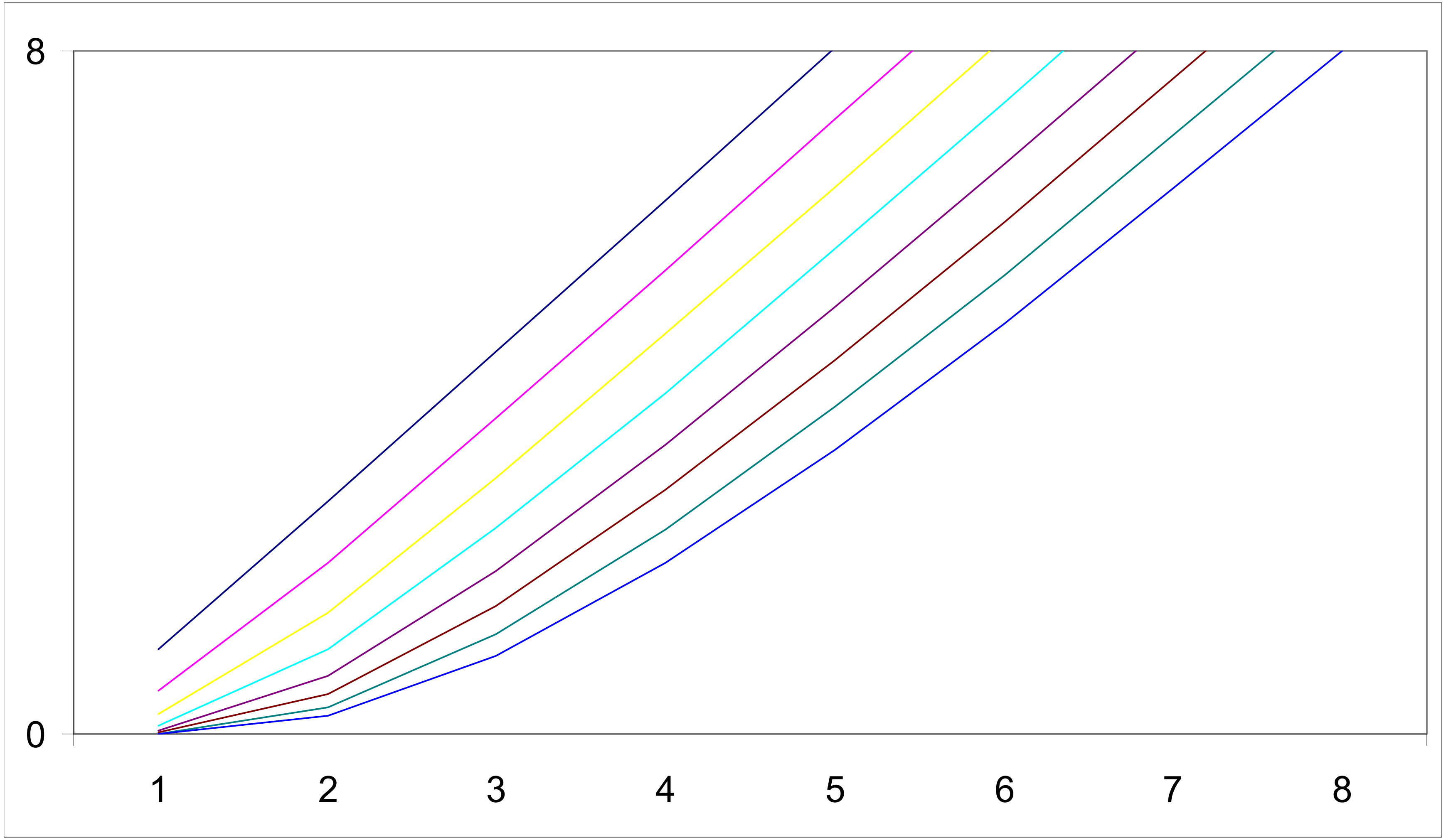
The 8×8 table of constants c(n, k)

n \ k	1	2	3	4	5	6	7	8
1	1	e	$e^{3/2}$	$e^{11/6}$	$e^{25/12}$	e^{H_5}	e^{H_6}	e^{H_7}
2	$\frac{1}{2}$	2	$\frac{1}{2}e^{2}$	$2e$	$\frac{1}{2}e^{8/3}$	$2e^{3/2}$	$\frac{1}{2}e^{46/15}$	$2e^{11/6}$
3	$\frac{1}{\sqrt{3}}e^{-\pi/\sqrt{12}}$	$\frac{1}{\sqrt{3}}e^{\pi/\sqrt{12}}$	3	$\frac{1}{\sqrt{3}}e^{3-\frac{\pi}{\sqrt{12}}}$	$\frac{1}{\sqrt{3}}e^{\frac{3}{2}+\frac{\pi}{\sqrt{12}}}$	$3e$	$\frac{1}{\sqrt{3}}e^{\frac{15}{4}-\frac{\pi}{\sqrt{12}}}$	$\frac{1}{\sqrt{3}}e^{\frac{21}{10}+\frac{\pi}{\sqrt{12}}}$
4	$\frac{1}{2}e^{-\pi/2}$	1	$\frac{1}{2}e^{\pi/2}$	4	$\frac{1}{2}e^{4-\frac{\pi}{2}}$	e^{2}	$\frac{1}{2}e^{\frac{4}{3}+\frac{\pi}{2}}$	$4e$
5	$\frac{1}{5^{1/4}\Phi^{\sqrt{5}/2}}e^{-\frac{\pi}{2}\sqrt{1+2/\sqrt{5}}}$	$\frac{\Phi^{\sqrt{5}/2}}{5^{1/4}}e^{-\frac{\pi}{2}\sqrt{1-2/\sqrt{5}}}$	$\frac{\Phi^{\sqrt{5}/2}}{5^{1/4}}e^{\frac{\pi}{2}\sqrt{1-2/\sqrt{5}}}$	$\frac{1}{5^{1/4}\Phi^{\sqrt{5}/2}}e^{\frac{\pi}{2}\sqrt{1+2/\sqrt{5}}}$	5	$5e^{\psi\left(\frac{6}{5}\right)+\gamma}$	$5e^{\psi\left(\frac{7}{5}\right)+\gamma}$	$5e^{\psi\left(\frac{8}{5}\right)+\gamma}$
6	$\frac{1}{\sqrt{12}}e^{-\frac{\pi}{2}\cdot\sqrt{3}}$	$\frac{2}{\sqrt{3}}e^{-\pi/\sqrt{12}}$	$3/2$	$\frac{2}{\sqrt{3}}e^{\pi/\sqrt{12}}$	$\frac{1}{\sqrt{12}}e^{\frac{\pi}{2}\cdot\sqrt{3}}$	6	$\frac{1}{\sqrt{12}}e^{6-\frac{\pi}{2}\cdot\sqrt{3}}$	$\frac{2}{\sqrt{3}}e^{3-\frac{\pi}{\sqrt{12}}}$
7	$7e^{\psi\left(\frac{1}{7}\right)+\gamma}$	$7e^{\psi\left(\frac{2}{7}\right)+\gamma}$	$7e^{\psi\left(\frac{3}{7}\right)+\gamma}$	$7e^{\psi\left(\frac{4}{7}\right)+\gamma}$	$7e^{\psi\left(\frac{5}{7}\right)+\gamma}$	$7e^{\psi\left(\frac{6}{7}\right)+\gamma}$	7	$7e^{\psi\left(\frac{8}{7}\right)+\gamma}$
8	$\frac{1}{2(1+\sqrt{2})^{\sqrt{2}}}e^{-\frac{\pi}{2}(1+\sqrt{2})}$	$e^{-\pi/2}$	$\frac{(1+\sqrt{2})^{\sqrt{2}}}{2}e^{-\frac{\pi}{2}(\sqrt{2}-1)}$	2	$\frac{(1+\sqrt{2})^{\sqrt{2}}}{2}e^{\frac{\pi}{2}(\sqrt{2}-1)}$	$e^{\pi/2}$	$\frac{1}{2(1+\sqrt{2})^{\sqrt{2}}}e^{\frac{\pi}{2}(1+\sqrt{2})}$	8

<u>The 8 × 8 table for c(*n* ,*k*) numerical values</u>

n \ k	1	2	3	4	5	6	7	8
1	1	2,71828183	4,48168907	6,25470095	8,031195	9,80932372	11,5883467	13,3679111
2	0,5	2	3,69452805	5,43656366	7,19595805	8,96337814	10,7351079	12,5094019
3	0,23311909	1,42988431	3	4,68232215	6,40829688	8,15484549	9,91247608	11,67667821
4	0,10393979	1	2,40523869	4	5,67492015	7,3890561	9,1246768	10,8731273
5	0,04494183	0,6874742	1,907959	3,392764	5	6,668	8,373	10,098
6	0,01900311	0,46623819	1,5	2,85976862	4,38524599	6	7,66640372	9,36464431
7	0,0079	0,312	1, 1704	2,397	3,83	5,379	7	8,662
8	0,00324104	0,20787958	0,909	2	3,3332	4,81047738	6,3766	8

The graphs of c(n, k) as functions of k, for n =1 up to 8 and k =1 up to 8



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